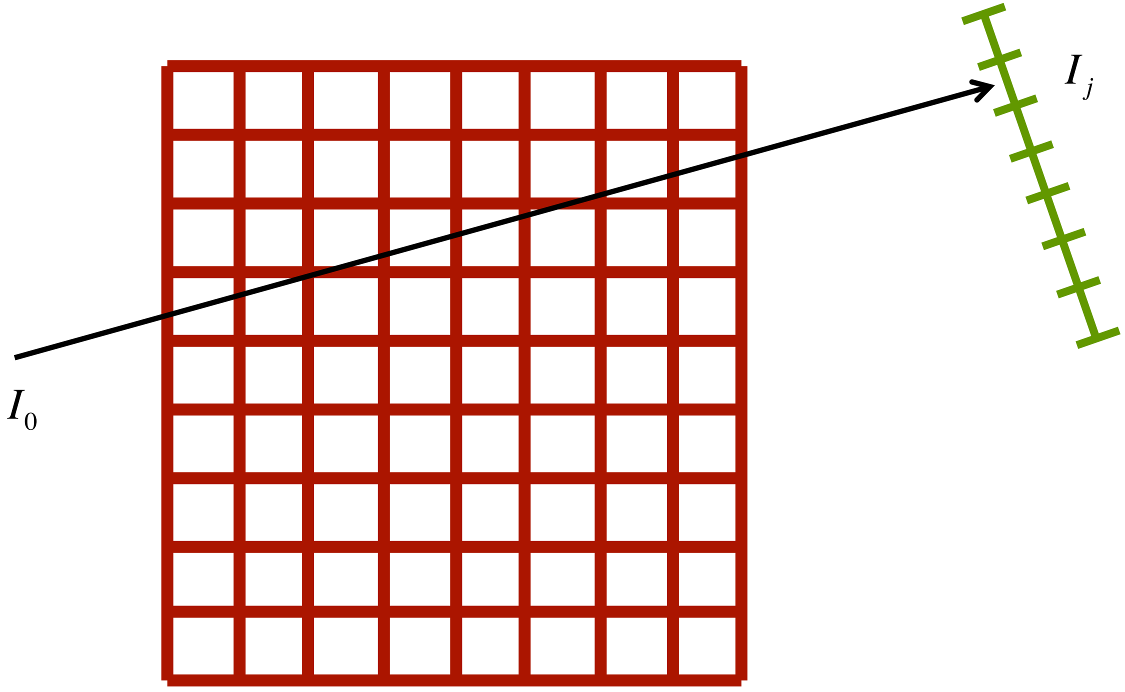
Iterative CT reconstruction

A short introduction



Analytic reconstruction

Previous lectures:

Analytic reconstruction techniques

- analytic description of the forward problem

$$\lambda(p,\phi) = \int_A f(\boldsymbol{x})\,\delta(p - \boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi)\;\mathrm{d}^2x \;= \Re\{f(\boldsymbol{x})\}$$

- solve by analytically inverting the Radon transform operator

$$f(\boldsymbol{x}) = \int_0^\pi \int_{-\infty}^{\infty} |\nu|\Lambda_\phi(\nu)\exp(2\pi i\,\nu\boldsymbol{x}\cdot\hat{\boldsymbol{n}}_\phi)\;\mathrm{d}\nu\,\mathrm{d}\phi$$

Algebraic reconstruction

Now:

Algebraic reconstruction techniques

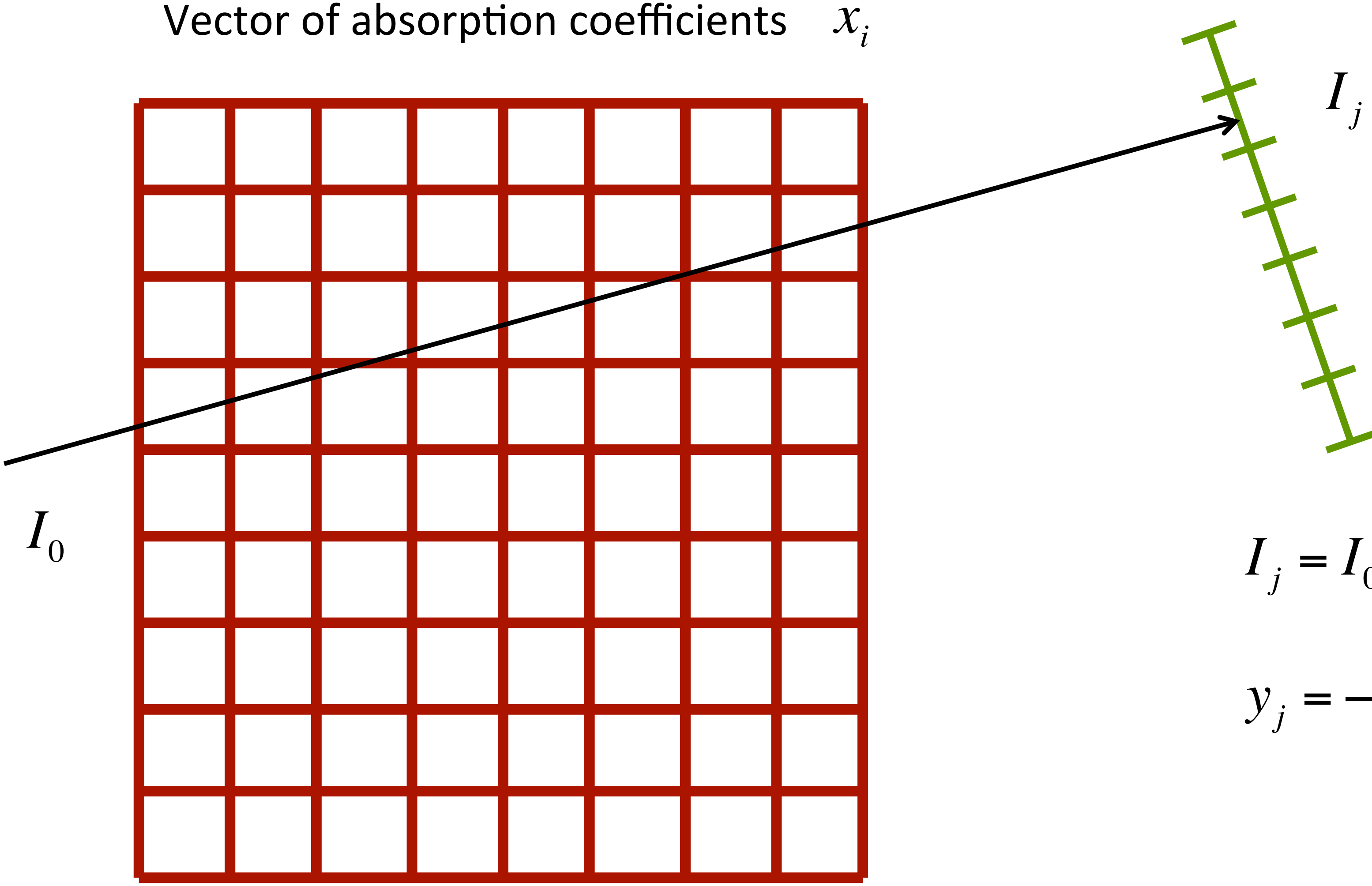
- discrete problem representation
  (through algebraic equations)
- optimization based reconstruction techniques

Problem formulation

Unknown image

Vector of absorption coefficients x_i

Observation



$$I_j = I_0 \exp\left(-y_j\right)$$

$$y_j = -\log\left(\frac{I_j}{I_0}\right)$$

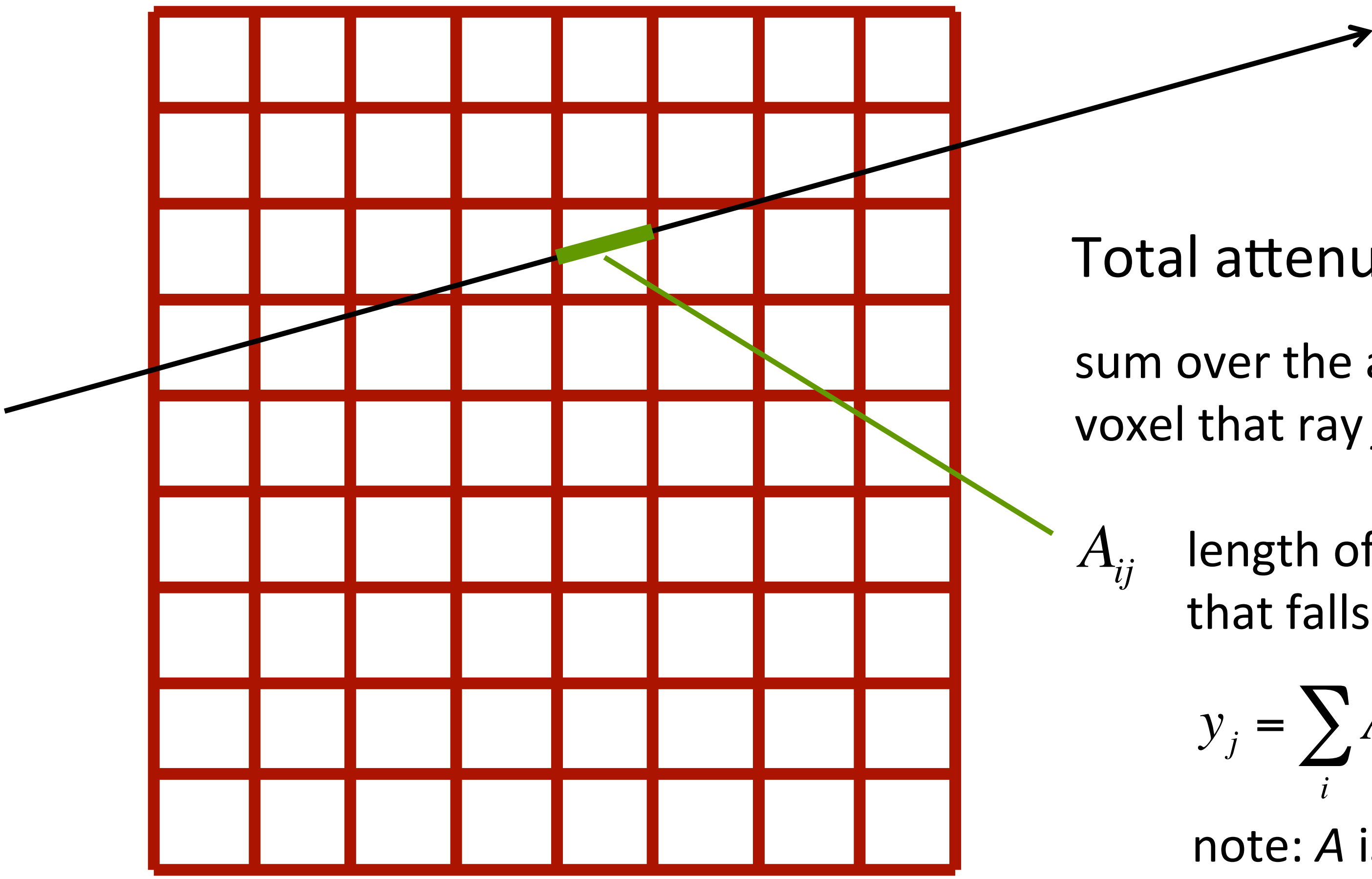
Problem formulation

Unknown image

Vector of attenuation coefficients x_i

Observation

y_j



Total attenuation y_j:

sum over the attenuation in every voxel that ray *j* passes through

A_{ij} length of the segment of ray *j* that falls into voxel *i*

$$y_j = \sum_i A_{ij} x_i$$

note: *A* is sparse

Problem formulation

- y_j is measured
- A_{ij} can be constructed (similar to ray tracing in dose calculation)

CT reconstruction task:

Find a solution to the linear system of equations

$$y_j = \sum_i A_{ij} x_i$$

Difficulties

1. One difficulty: Problem size

$$y_j = \sum_i A_{ij} x_i$$

(#pixels)^2 x #angles

512 x 512 x 200

A_{ij} is sparse but still very large (may not fit in memory)

2. Problem is ill-posed

- underdetermined (many solutions) (reconstruction from undersampled projections)
- overdetermined (no solution)

Projection methods

First approach:

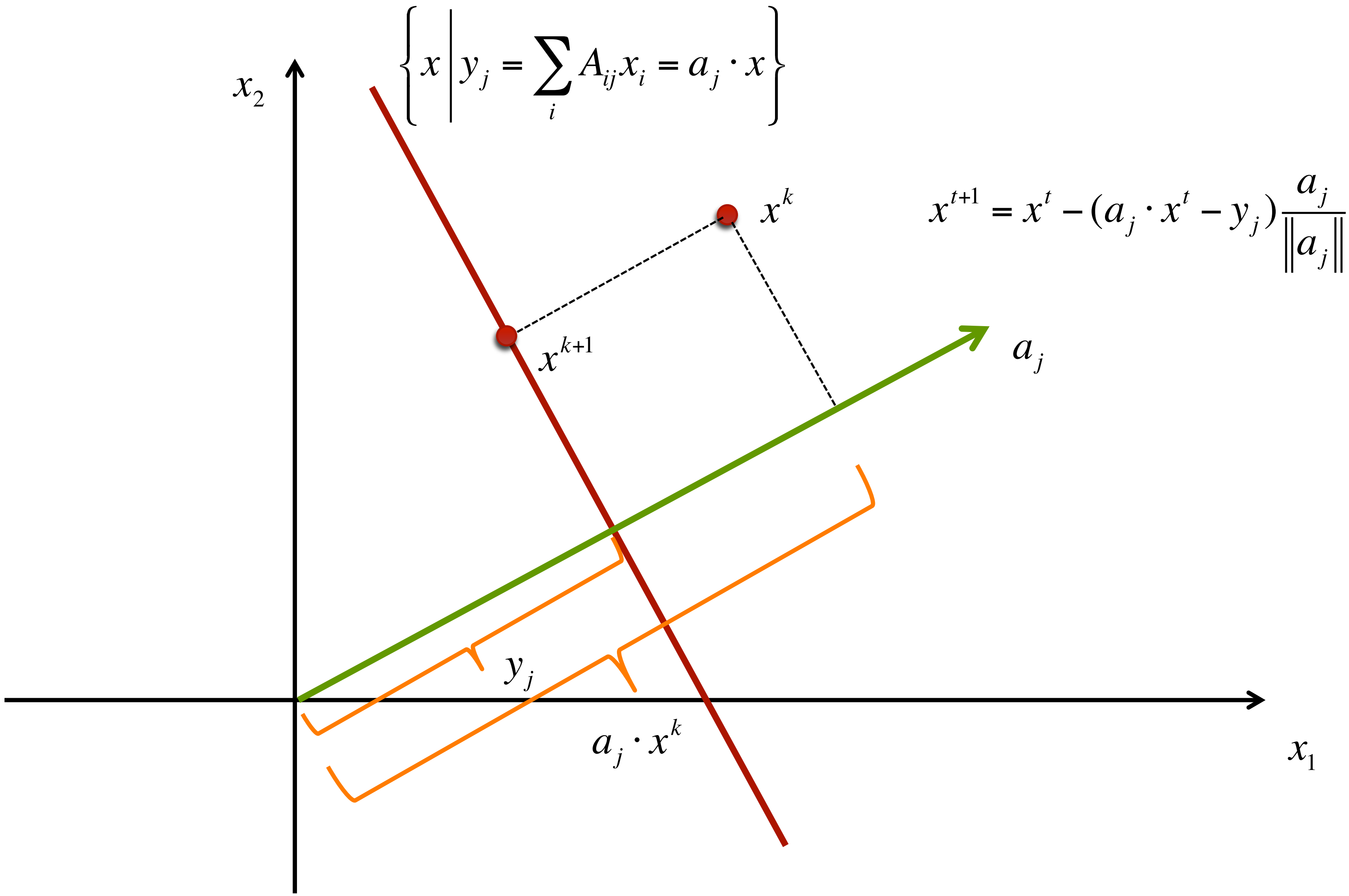
Find a solution to the system of linear equations

- x is a N-dimensional vector
- each measurement y_j defines in (N-1)-dimensional hyper-plane

Solution method:

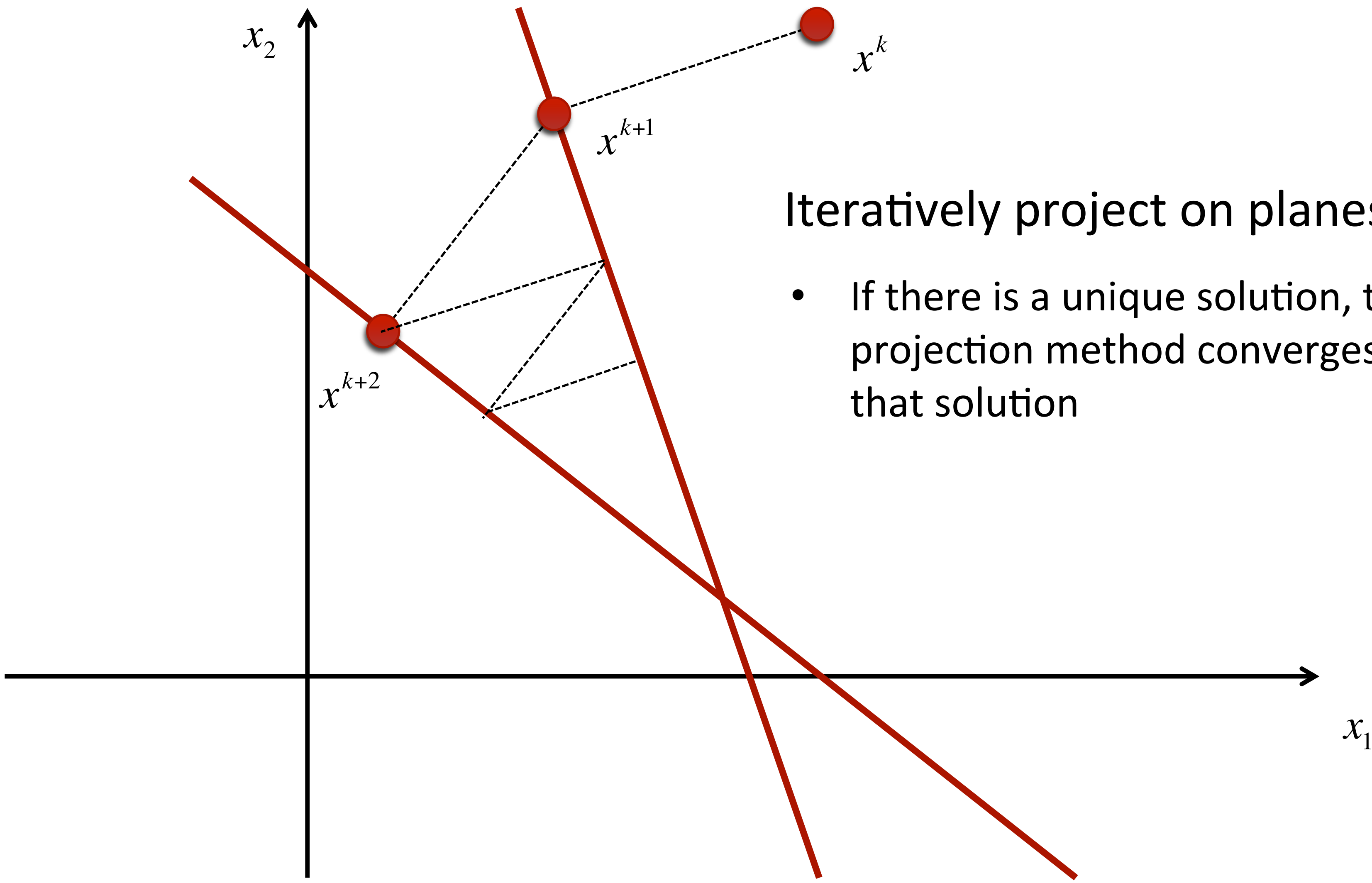
Iteratively project on hyper-planes

Projection



$\left\{ x \,\middle|\, y_j = \sum_i A_{ij} x_i = a_j \cdot x \right\}$
$x^{t+1} = x^t - (a_j \cdot x^t - y_j)\frac{a_j}{\|a_j\|}$
x_2
x^k
x^{k+1}
a_j
y_j
$a_j \cdot x^k$
x_1

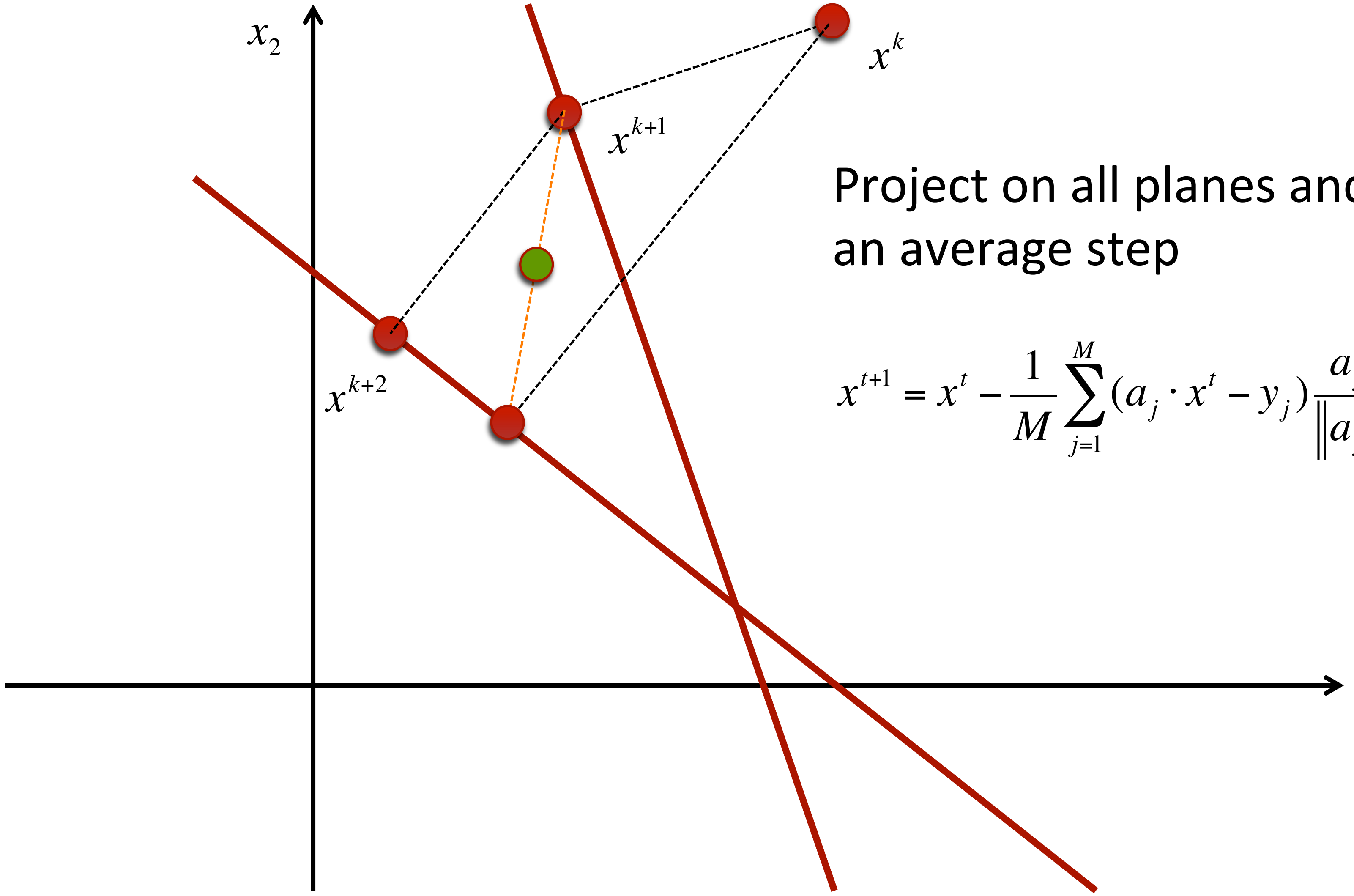
Iterative Projection



Iteratively project on planes

- If there is a unique solution, the projection method converges to that solution

Variation



Project on all planes and take an average step

$$x^{t+1} = x^t - \frac{1}{M}\sum_{j=1}^{M}(a_j \cdot x^t - y_j)\frac{a_j}{\|a_j\|}$$

Optimization methods

Second approach:

Look at this as an optimization problem

(can address the ill-posed-ness this way)

$$y_j = \sum_i A_{ij} x_i$$ System of linear equations

$$\Downarrow$$

$$\underset{x}{\text{minimize}} \quad \sum_j \left(y_j - \sum_i A_{ij} x_i \right)^2$$ Least square minimization problem

Gradient descent

Most basic optimization method: Gradient descent

$$C(x) = \sum_j \left(y_j - \sum_i A_{ij} x_i \right)^2$$

Gradient vector points in the direction of steepest ascent

$$\nabla_x C = \begin{pmatrix} \dfrac{\partial C}{\partial x_1} \\ \vdots \\ \dfrac{\partial C}{\partial x_n} \end{pmatrix}$$

$$x^{t+1} \leftarrow x^t - \alpha \nabla_x C(x^t)$$

Gradient descent

Objective function

$$C(x)=\sum_j\left(y_j-\sum_i A_{ij}x_i\right)^2$$

Calculate gradient

$$\frac{\partial C}{\partial x_i}=-\sum_j 2\left(y_j-\sum_i A_{ij}x_i\right)A_{ij}$$

$$x^{t+1}=x^t-(a_j\cdot x^t-y_j)\frac{a_j}{\left\|a_j\right\|}$$

$$\nabla_x C=-\sum_j 2\left(y_j-\sum_i A_{ij}x_i\right)a_j$$

$$=-\sum_j 2\left(y_j-a_j\cdot x\right)a_j$$

Calculate descent

$$x^{t+1}=x^t-\alpha\nabla_x C(x^t)$$

Gradient descent

Compare to projection method

Projection

$$x^{t+1} = x^t - \frac{1}{M}\sum_{j=1}^{M}(a_j \cdot x^t - y_j)\frac{a_j}{\|a_j\|}$$

Calculate descent

$$x^{t+1} = x^t - \alpha \nabla_x C(x^t)$$

$$= x^t - \alpha \sum_{j=1}^{M} 2\left(a_j \cdot x^t - y_j\right) a_j$$

- Projection method can be interpreted as gradient descent (for particular step size and weighting factors)

Summary

Historically: Different versions of projection methods (ART, SIRT, SART, …)

Contemporary topics:

- Reconstruction from incomplete data, Compressed sensing methods

$$\underset{x}{\text{minimize}} \quad \sum_j \left(y_j - \sum_i A_{ij} x_i \right)^2 + \mathrm{TV}(x)$$

L_1-regularization

- Reconstruction using prior knowledge
- Improved forward models, e.g. beam hardening