Image Reconstruction 2b – Fully 3D Reconstruction

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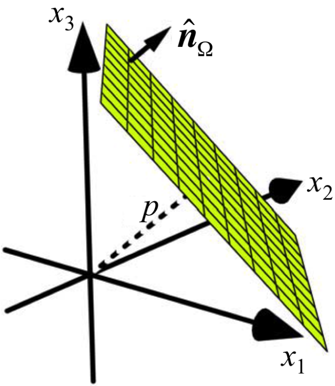
Outline

1. The 3D Radon transform and its inverse
   - The 3D Radon transform
   - Inversion of the 3D Radon transform
2. Getting 3D Radon transform from cone beam data (1990s)
   - Grangeat's trick
3. Tuy theorem
4. Helical scanning
5. The Katsevich breakthrough (2002)

3D Radon transform



- The 3D Radon transform of $f(\boldsymbol{x})$ is the integral of $f(\boldsymbol{x})$ over 2D planes perpendicular to $\hat{\boldsymbol{n}}_\Omega$

$$\mathfrak{R}f(p, \hat{\boldsymbol{n}}_\Omega) = \int_V f(\boldsymbol{x})\, \delta(p - \boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\Omega)\ \mathrm{d}^3 x$$

1. Note: The δ-function "picks" those points $\boldsymbol{x}$ that lie on the plane shown (plane at distance p from origin).
2. Note: In 3D (and higher dimensions) the Radon transform differs from the "x-ray transform" (integration over lines).

The inverse 3D Radon transform

Inversion of the Radon transform in 3D is beautifully simple:

$$f(\boldsymbol{x}) = -\frac{1}{8\pi^2} \int\limits_{4\pi} \mathfrak{R}'' f(\boldsymbol{x} \cdot \hat{\boldsymbol{n}}_\Omega, \hat{\boldsymbol{n}}_\Omega)\, \mathrm{d}\Omega,$$

where

$$\mathfrak{R}'' f(p, \hat{\boldsymbol{n}}_\Omega) = \frac{\partial^2 \mathfrak{R}'' f(p, \hat{\boldsymbol{n}}_\Omega)}{\partial p^2}.$$

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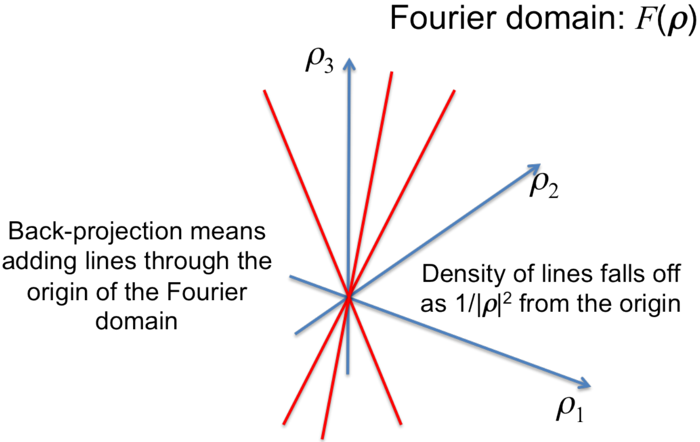
Steps involved:

1. take second derivative (with respect to p) of Radon transform,
2. back-project over planes containing $\boldsymbol{x}$,
3. integrate over all angles.

Intuitive "derivation" of inverse 3D Radon transform

- Central slice theorem in 3D: every back-projection ("smearing out" over planes) for a given orientation $\hat{\boldsymbol{n}}_\Omega$ corresponds with adding a line through the origin of the Fourier domain.
- The density of lines falls off as $1/\boldsymbol{\rho}^2$.
- To compensate for the low-pass effect, we must multiply with $\boldsymbol{\rho}^2$ (compare with the $|\boldsymbol{\rho}| = |\nu|$ filter in the 2D case).
- Multiplication with $\boldsymbol{\rho}^2$ in the Fourier domain corresponds with taking the second derivative in the spatial domain.

Intuitive "derivation" of inverse 3D Radon transform



So what is the problem?

- The problem is that with x-rays we can only measure line integrals (along the rays), not the planar integrals needed for the 3D Radon transform.

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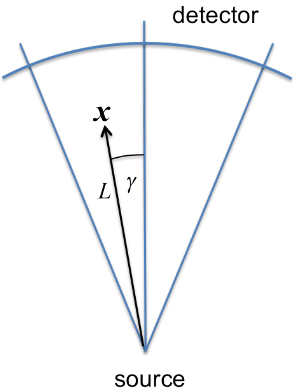
Getting 3D Radon transform from cone beam data

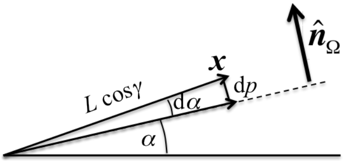
- The problem is that with x-rays we can only measure line integrals (along the rays), not the planar integrals needed for the 3D Radon transform.
- But, can't we just integrate line integrals across the plane to get the planar integral?

Getting 3D Radon transform from cone beam data

Top view



Side view



$$dp = L\cos\gamma\, d\alpha$$

Getting 3D Radon transform from cone beam data

Measured line integrals:

$$\int_0^\infty f\left(\boldsymbol{x}(L,\gamma,\alpha)\right)\mathrm{d}L$$

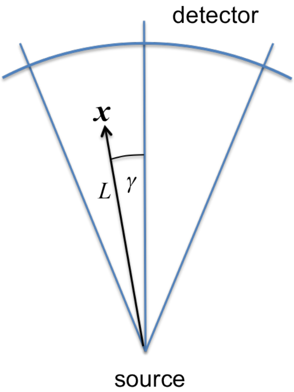
Integrate over the plane:

$$\int_{-\pi}^{\pi}\int_0^\infty f\left(\boldsymbol{x}(L,\gamma,\alpha)\right)\mathrm{d}L\,\mathrm{d}\gamma = \int_{-\pi}^{\pi}\int_0^\infty \frac{1}{L} f\left(\boldsymbol{x}(L,\gamma,\alpha)\right)L\,\mathrm{d}L\,\mathrm{d}\gamma$$
$$\neq \mathfrak{R}f(p,\hat{\boldsymbol{n}}_\Omega)$$

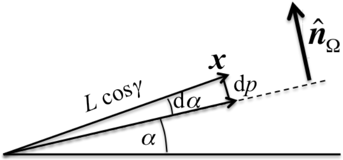
This would give the integral of $\frac{1}{L}f$ over the plane.

Grangeat's trick

Top view



Side view



$$\mathrm{d}p = L \cos\gamma \, \mathrm{d}\alpha$$

Grangeat's "trick"

- Grangeat's idea: Compensate for incorrect integration over the plane by taking an incorrect derivative of the Radon transform with respect to p.

Apply a weight factor of $1/\cos\gamma$ and take the derivative with respect to the tilt angle, α:

$$\int_{-\pi}^{\pi} \frac{\partial}{\partial\alpha} \frac{1}{\cos\gamma} \int_0^{\infty} f\left(\boldsymbol{x}(L,\gamma,\alpha)\right) \mathrm{d}L\,\mathrm{d}\gamma = \int_{-\pi}^{\pi} \frac{\partial}{\partial p} \frac{L\cos\gamma}{\cos\gamma} \int_0^{\infty} f\left(\boldsymbol{x}(L,\gamma,\alpha)\right) \mathrm{d}L\,\mathrm{d}\gamma$$

$$= \frac{\partial}{\partial p} \int_{-\pi}^{\pi} \int_0^{\infty} f\left(\boldsymbol{x}(L,\gamma,p)\right) L\,\mathrm{d}L\,\mathrm{d}\gamma$$

$$= \frac{\partial \mathfrak{R} f(p, \hat{\boldsymbol{n}}_\Omega)}{\partial p}.$$

Problems with Grangeat-type methods

- "Long object" problem – solvable, approximately
- Numerical instabilities

Tuy's sufficiency condition

Theorem (Tuy)

Any plane through an object point x must cut the source trajectory.[a]

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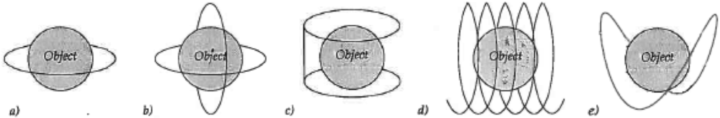
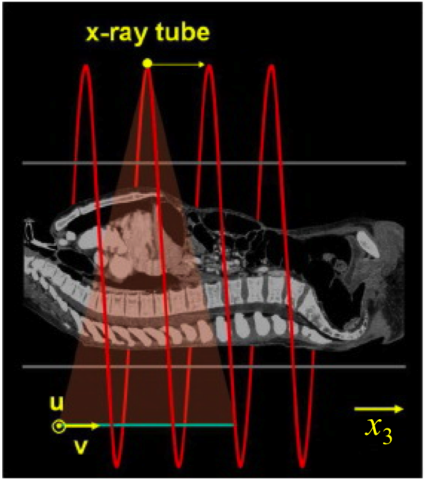
[a] H.K.Tuy, An inversion formula for cone-beam reconstruction, SIAM J Appl. Math., 43(3), 546-552, 1983. See also A.A. Kirillow, On a problem of I. M. Gel'fand, Soviet Math., 2 (1961), pp. 268-269.

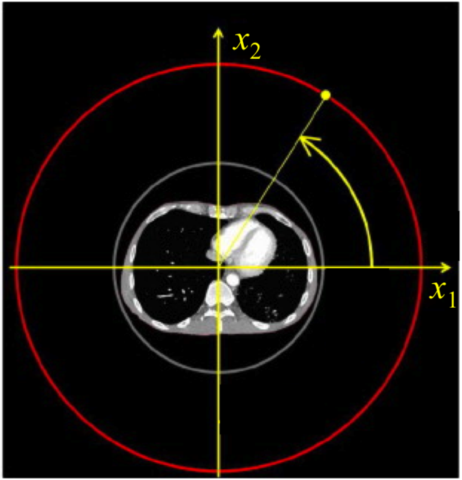
- Condition for trajectory to facilitate exact reconstruction

Examples of source trajectories – which ones fulfill the Tuy condition?

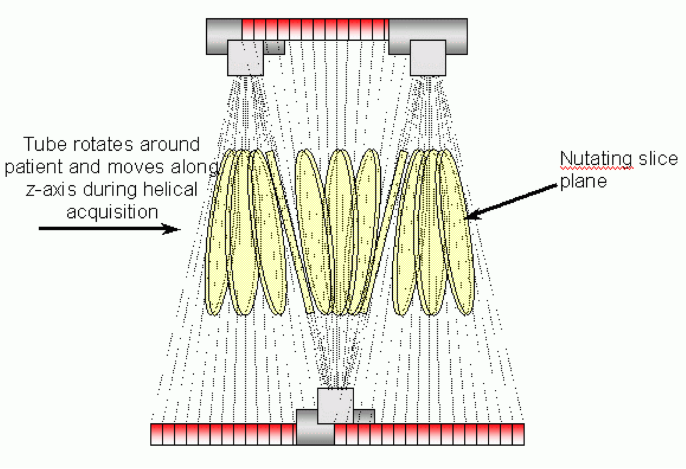


Helical scanning geometry





Helical scanning geometry: Resort into planes



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The Katsevich breakthrough

- Filtered backprojection algorithm for cone beam data
- Exact reconstruction (aside from discretization)
- Good for "long objects" (projection data only needed near the ROI)

The Katsevich algorithm

Detector signal and its derivative:

$$D(\boldsymbol{y}(s), \boldsymbol{\theta})$$

$$D'(\boldsymbol{y}(s), \boldsymbol{\theta}) = \left.\frac{\partial D(\boldsymbol{y}(q), \boldsymbol{\theta})}{\partial q}\right|_{q=s}$$

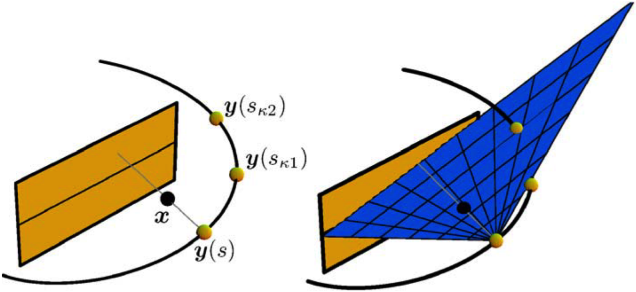
Further definitions:

$$\boldsymbol{b} = \frac{\boldsymbol{x} - \boldsymbol{y}(s)}{|\boldsymbol{x} - \boldsymbol{y}(s)|}; \quad \boldsymbol{y}(s)\text{: source position}$$

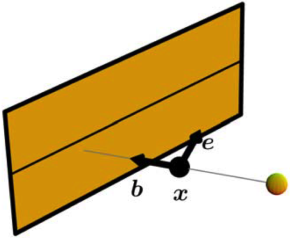
$\boldsymbol{e}$: perpendicular to $\boldsymbol{b}$, must be appropriately chosen

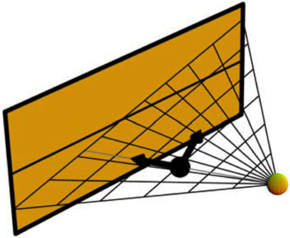
$\boldsymbol{b}$ and $\boldsymbol{e}$ span the κ plane for filtering

The Kappa plane



The Kappa plane





The Katsevich algorithm

$$f(\boldsymbol{x}) = -\frac{1}{2\pi^2} \int\limits_{I_{\mathsf{PI}}(\boldsymbol{x})} \frac{I(s,\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}(s)|}\,\mathrm{d}s$$

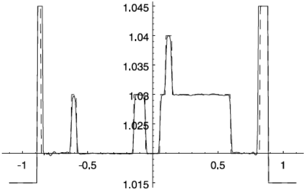
where

$$I(s,\boldsymbol{x}) = \int\limits_{-\pi}^{\pi} \frac{1}{\sin\gamma} D'(\boldsymbol{y}(s), \cos\gamma\,\boldsymbol{b} + \sin\gamma\,\boldsymbol{e})\,\mathrm{d}\gamma$$

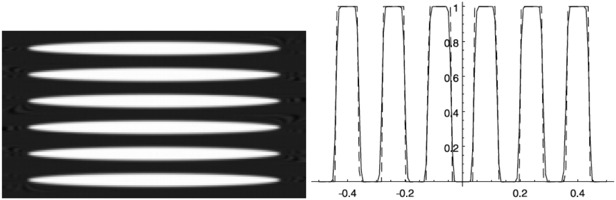
Note:

- $1/\sin\gamma$ is the filter, it corresponds to a “Hilbert” filter.
- Backprojection is done over the PI (parametric interval), see Danielsson et al.

Results: 3D Shepp phantom – from Katsevich 2002



Results: Defrise slice phantom – from Katsevich 2002



Further reading on fully 3D reconstruction

- **A. Katsevich**: *Theoretically exact filtered backprojection-type inversion algorithm for spiral CT*. SIAM J. Appl. Math. 62(6):2012-2026, 2002.
- **A. Katsevich**: *Analysis of an exact inversion algorithm for spiral cone-beam CT*. Phys. Med. Biol. 47:2583-2597, 2002.
- **C. Bontus, T. Köhler**: *Reconstruction Algorithms for Computed Tomography*. Advances in Imaging and Electron Physics 151:1-63, 2008.
- Papers by P.-E. Danielsson et al., Linköping, Sweden
- **F. Natterer**: *The Mathematics of Computerized Tomography*. Reprint: SIAM Classics in Applied Mathematics, 2001.
- **H.K. Tuy**: *An inversion formula for cone-beam reconstruction*. SIAM J Appl. Math., 43(3), 546-552, 1983.