Image Reconstruction 2a – Cone Beam Reconstruction

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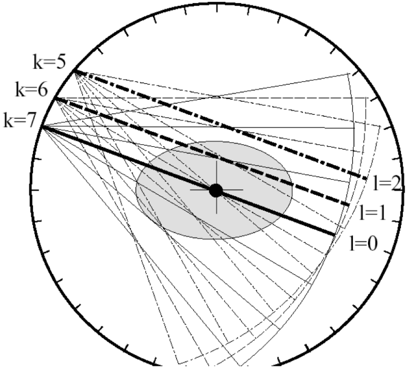
Outline

1. Reconstruction from fan projections
   - Resorting fan to parallel projections
   - Weighted filtered back-projection for fan beams

2. Cone-beam reconstruction
   - The Feldkamp algorithm
   - Limitations of the Feldkamp algorithm

Resorting fan $\rightsquigarrow$ parallel



- k: counter of source positions
- l: counter of projection lines within each fan-projection
- $k + l = 7 =$ const. yields parallel projections (not equally spaced)

Weighted filtered back-projection for fan beams

We start with the reconstruction formula (note ϕ integration from 0 to 2π):

$$f(\boldsymbol{x}) = \frac{1}{2} \int_{0}^{2\pi} \int_{-\infty}^{\infty} \lambda_\phi(p)\, h^{-1}(\boldsymbol{x} \cdot \hat{\mathbf{n}}_\phi - p)\, \mathrm{d}p\, \mathrm{d}\phi$$

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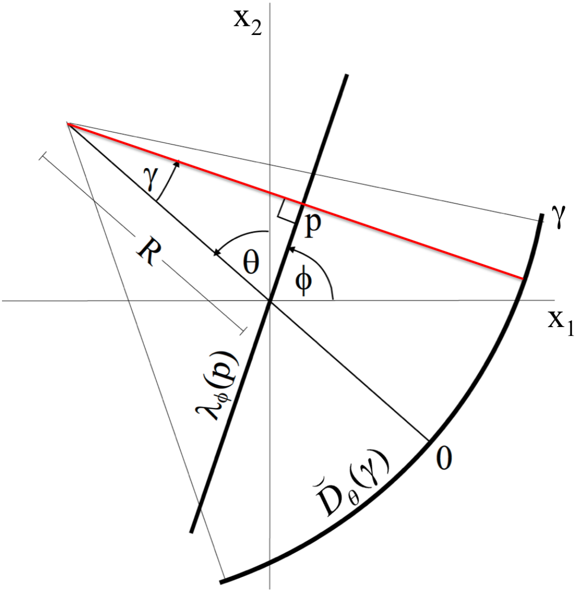
Now introduce fan beam coordinates γ, θ such that:

$$p = R \sin(\gamma)$$
$$\phi = \theta + \gamma$$



$$p = R\sin(\gamma)$$
$$\phi = \theta + \gamma$$

Note: the red projection line is the same line once expressed as $\lambda_\phi(p)$ and then as $\breve{D}_\theta(\gamma)$.

Weighted filtered back-projection for fan beams, cont'd

With the Jacobian determinant $\mathrm{d}p\,\mathrm{d}\phi = R\cos\gamma\,\mathrm{d}\gamma\,\mathrm{d}\theta$ we obtain:

$$f(\boldsymbol{x}) = \frac{1}{2}\int\limits_{0}^{2\pi}\int\limits_{-\pi/2}^{\pi/2} \breve{D}_\theta(\gamma)\,R\cos\gamma\,h^{-1}(\boldsymbol{x}\cdot\hat{\mathbf{n}}_{\theta+\gamma} - R\sin\gamma)\,\mathrm{d}\gamma\,\mathrm{d}\theta.$$

Weighted filtered back-projection for fan beams, cont'd

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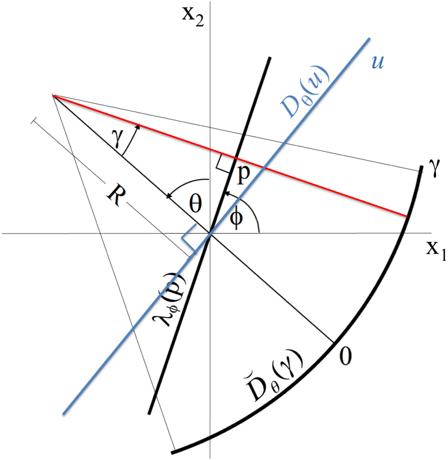
Now define

$\gamma_{\theta,x}$:= angle between line (source — $\boldsymbol{x}$) and central ray

$L_{\theta,x}$:= distance between source and point $\boldsymbol{x}$

$$f(\boldsymbol{x}) = \frac{1}{2}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \breve{D}_\theta(\gamma)\,R\cos\gamma\,h^{-1}\,(L_{\theta,x}\sin(\gamma_{\theta,x} - \gamma))\,\mathrm{d}\gamma\,\mathrm{d}\theta.$$

Straight detector



The blue line represents the straight detector, virtually positioned at the isocenter (origin)

$$\gamma = \arctan\left(\frac{u}{R}\right)$$

Weighted filtered back-projection for fan beams, cont'd

Now: **straight detector** (projected into the origin) measuring $D_\theta(u)$ such that $\gamma = \arctan\left(\frac{u}{R}\right)$ and $\mathrm{d}\gamma = \frac{R}{R^2+u^2}\,\mathrm{d}u$.

Weighted filtered back-projection for fan beams, cont'd

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Using the angle addition formula for the sine we obtain:

$$f(\boldsymbol{x}) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} D_\theta(u)\,\frac{R^3}{(R^2+u^2)^{3/2}}\,h^{-1}\left(\frac{L_{\theta,x}R(u_{\theta,x}-u)}{\sqrt{R^2+u_{\theta,x}^2}\sqrt{R^2+u^2}}\right)\mathrm{d}u\;\mathrm{d}\theta.$$

Weighted filtered back-projection for fan beams, cont'd

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Now define $W_{\theta,x} := L_{\theta,x}/\sqrt{R^2+u_{\theta,x}^2}$. Note that $W_{\theta,x}$ equals the ratio of the projection of point $\boldsymbol{x}$ onto the central ray to R.

$$f(\boldsymbol{x}) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} D_\theta(u)\,\frac{R^3}{(R^2+u^2)^{3/2}}\,h^{-1}\left(\frac{W_{\theta,x}R}{\sqrt{R^2+u^2}}(u_{\theta,x}-u)\right)\mathrm{d}u\;\mathrm{d}\theta.$$

Weighted filtered back-projection for fan beams, cont'd

Finally, we make use of the relationship $h^{-1}(\alpha x) = \frac{1}{\alpha^2} h^{-1}(x)$. This follows from the representation of h^{-1} as the inverse Fourier transform of $H^{-1} = |\nu|$:

$$f(\boldsymbol{x}) = \frac{1}{2} \int_0^{2\pi} \frac{1}{W_{\theta,x}^2} \int_{-\infty}^{\infty} D_\theta(u) \, \frac{R}{\sqrt{R^2 + u^2}} \, h^{-1} \left(u_{\theta,x} - u \right) \mathrm{d}u \; \mathrm{d}\theta.$$

Weighted filtered back-projection for fan beams, algorithm

1. Calculate modified projections from the Detector signal $D_\theta(u)$ by multiplying with $\frac{R}{\sqrt{R^2+u^2}}$.

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2. Convolve modified fan-beam projections with h^{-1}. The Ram-Lak or Shepp-Logan filter derived above for parallel data can be used here as well.

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3. Perform **weighted** filtered back-projections along the fan using $1/W_{\theta,x}^2$ as the weight factor. Remember: $W_{\theta,x}$ equals the ratio of the projection of point $\boldsymbol{x}$ onto the central ray to R.

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4. Integrate (sum up) back-projections from all fans.

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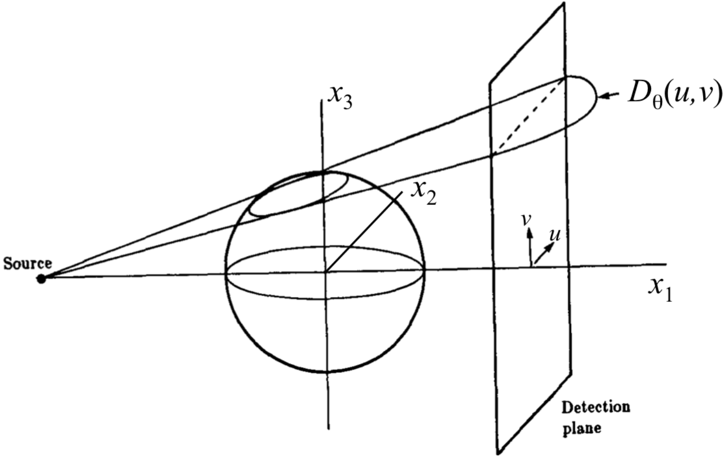
   - The Feldkamp algorithm
   - Limitations of the Feldkamp algorithm

Feldkamp algorithm

Idea: Reconstruct a 3D object from tilted fans. Treat tilted fans in the same way as in 2D geometry.



Feldkamp algorithm

Two important differences (due to fan tilting):

- The source distance is slightly **enlarged** for the tilted fans: $\tilde{R} = \sqrt{R^2 + v^2}$.
- The angular increment $\mathrm{d}\tilde{\theta}$ is slightly **decreased**: $\mathrm{d}\tilde{\theta}\tilde{R} = \mathrm{d}\theta R$.

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Then the "reconstruction" formula becomes:

$$f(\boldsymbol{x}) = \frac{1}{2}\int_0^{2\pi} \frac{1}{W_{\theta,x}^2} \int_{-\infty}^{\infty} D_\theta(u,v)\, \frac{R}{\sqrt{R^2+u^2+v^2}}\, h^{-1}\left(u_{\theta,x} - u\right) \mathrm{d}u\ \mathrm{d}\theta.$$

Note that this is a **heuristic**, not an exact reconstruction formula!

Feldkamp algorithm

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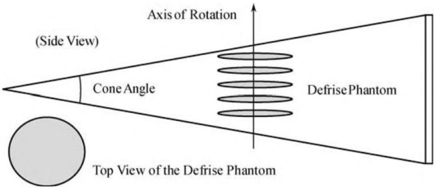
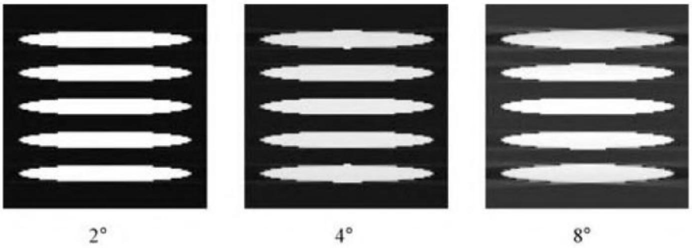
3. Perform weighted filtered back-projections along the **tilted** fans using $1/W_{\theta,x}^2$ as the weight factor. Note: $W_{\theta,x}$ is the ratio of the projection of point $\boldsymbol{x}$ onto the central ray (at $u=v=0$) of the cone, to R.

Feldkamp algorithm

1. Calculate modified projections from the Detector signal $D_\theta(u,v)$ by multiplying with $\frac{R}{\sqrt{R^2+u^2+v^2}}$.
2. Convolve modified projections with h^{-1} in the u direction. The Ram-Lak or Shepp-Logan filter derived above for parallel data can be used here as well.
3. Perform weighted filtered back-projections along the **tilted** fans using $1/W_{\theta,x}^2$ as the weight factor. Note: $W_{\theta,x}$ is the ratio of the projection of point $\boldsymbol{x}$ onto the central ray (at $u=v=0$) of the cone, to R.
4. Integrate (sum up) back-projections from all cones.

Limitations of the Feldkamp algorithm: Defrise phantom





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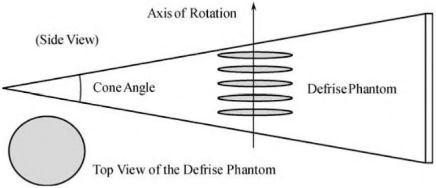
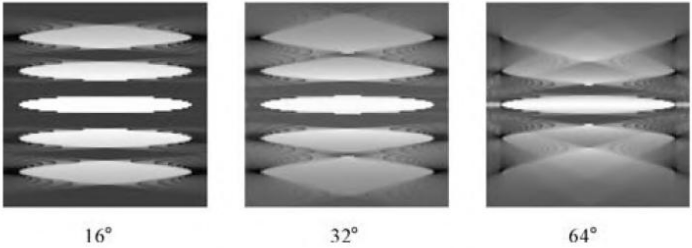


Artifacts occur at cone angles above $10°$:



Features of the Feldkamp algorithm

The Feldkamp algorithm is exact in the following sense:

1. For an object that has no contrast (no variation of density) in the z $(= x_3)$ direction, the reconstruction will be exact.
2. The Feldkamp algorithm produces the correct integral of the image intensity in the z $(= x_3)$ direction.

Homework 2

Homework 2:

a) Prove that the Feldkamp algorithm yields the correct result if the object has no density variation in the z $(= x_3)$ direction. You may do this either analytically or numerically.

- Note: Assume that the planar fan beam reconstruction for the same object yields the exact result.

Further Reading

- **L.A. Feldkamp, L.C. Davis, J.W. Kress**: *Practical cone-beam algorithm*. J. Opt. Soc. Am. A 1(6):612-619, 1984
- **A.C. Kak, M. Slaney**: *Principles of Computerized Tomographic Imaging*. Reprint: SIAM Classics in Applied Mathematics, 2001. PDF available: `http://www.slaney.org/pct/pct-toc.html`
- **F. Natterer**: *The Mathematics of Computerized Tomography*. Reprint: SIAM Classics in Applied Mathematics, 2001.
- **A.M. Cormack**: *Early Two-Dimensional Reconstruction and Recent Topics Stemming from it*. Nobel lecture, 1979. `http://www.nobelprize.org/nobel_prizes/medicine/laureates/1979/cormack-lecture.pdf`

Further Reading

- **R.N. Bracewell**: *The Fourier Transform and its Applications*. McGraw-Hill, New York, 3rd edition, revised, 1999.
- **T. Bortfeld**: *Röntgencomputertomographie: Mathematische Grundlagen*. In: Schlegel W, Bille J, eds. Medizinische Physik 2 (Medizinische Strahlenphysik). Heidelberg: Springer; 2002: 229-245. English translation available from author.
- **J. Radon**: *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*. Berichte der Sächsischen Akademie der Wissenschaften – Math.-Phys. Klasse, 69:262–277, 1917.