MASSACHUSETTS
GENERAL HOSPITAL

RADIATION ONCOLOGY

Deformable Image Registration
Part 3

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Review

- Warping based on “pulling”
- Quadratic image cost function (SSD)
- Use of image gradient to find correspondences
- Stabilized step length
- Demons algorithm
- How to use Jacobian matrix
- B-spline methods

Review

- What are the salient properties of the demons and B-spline method?

PDE methods

- Based on physical models
  - Especially linear elastic

$$\mu \nabla^2 u + (\mu + \lambda) \nabla \nabla \cdot u + F = 0$$

PDE methods

- Review of multi-variate calculus (in 3D)

$$\begin{array}{c} \nabla \; g \\ \mathrm{grad}\, g \end{array} \qquad \begin{bmatrix} \frac{dg}{dx} & \frac{dg}{dy} & \frac{dg}{dz} \end{bmatrix}$$

The gradient applied to a scalar function.
For a vector function, gradient becomes Jacobian.

PDE methods

- Review of multi-variate calculus (in 3D)

$$\nabla \cdot G \qquad \frac{dG_x}{dx} + \frac{dG_y}{dy} + \frac{dG_z}{dz}$$

$$\mathrm{div}\; g$$

The divergence (“flux density”) applied to
a vector field. It is the rate at which a fluid leaves
each point.

PDE methods

- Review of multi-variate calculus (in 3D)

$$\nabla \cdot \nabla \ g$$

$$\nabla^2 g$$

$$\Delta g$$

$$\text{div grad}\, g$$

$$\frac{d^2 g}{dx^2} + \frac{d^2 g}{dy^2} + \frac{d^2 g}{dz^2}$$

The Laplacian applied to a scalar field.
It is the rate at which the average within a region changes as the region grows.

PDE methods

- Review of multi-variate calculus (in 3D)

$$\nabla \times G \qquad \left[\left(\frac{dG_z}{dy} - \frac{dG_y}{dz}\right) \left(\frac{dG_x}{dz} - \frac{dG_z}{dx}\right) \left(\frac{dG_y}{dx} - \frac{dG_x}{dy}\right)\right]$$

$$\operatorname{curl} G$$

The curl ("circulation density") is applied to a vector field. It is a vector representing the strength and direction of circulation

PDE methods

- Review of multi-variate calculus (in 3D)

$$\nabla \cdot \nabla \; G$$
$$\nabla^2 G$$
$$\Delta G$$
$$\operatorname{div} \operatorname{grad} G$$

$$\left[\nabla^2 G_x \quad \nabla^2 G_y \quad \nabla^2 G_z\right]$$

The Vector Laplacian is equivalent to the scalar Laplaian, applied to each component independently.

PDE methods

- Linear elastic equation

$$\underbrace{\mu \nabla^2 u + (\mu + \lambda) \nabla \nabla \cdot u}_{\text{Internal force}} + F = 0$$

External force

External forces are balanced with internal forces

PDE methods

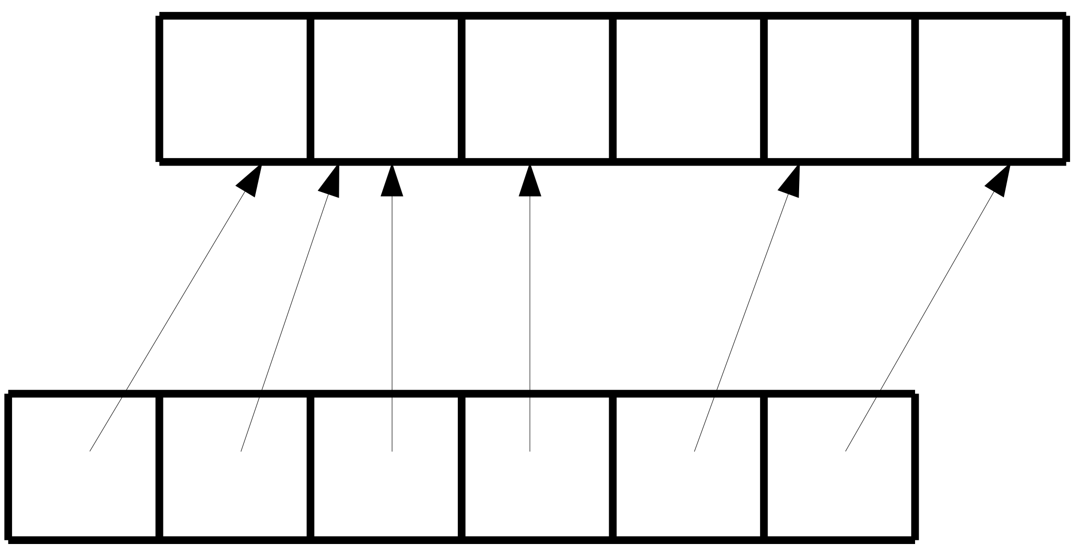
- Linear elastic equation

$$\mu \underbrace{\nabla^2 u}_{\substack{\text{Vector Laplacian =} \\ \text{Second derivative}}} + (\mu + \lambda) \nabla \nabla \cdot u + F = 0$$

External forces are balanced with internal forces

PDE methods

- Example in 1D



$$u = x^2$$

$$\nabla^2 u = 2$$

PDE methods

- Linear elastic equation

$$\mu \nabla^2 u + (\mu + \lambda) \underbrace{\nabla \nabla \cdot}_{\text{Gradient of divergence}} u + F = 0$$

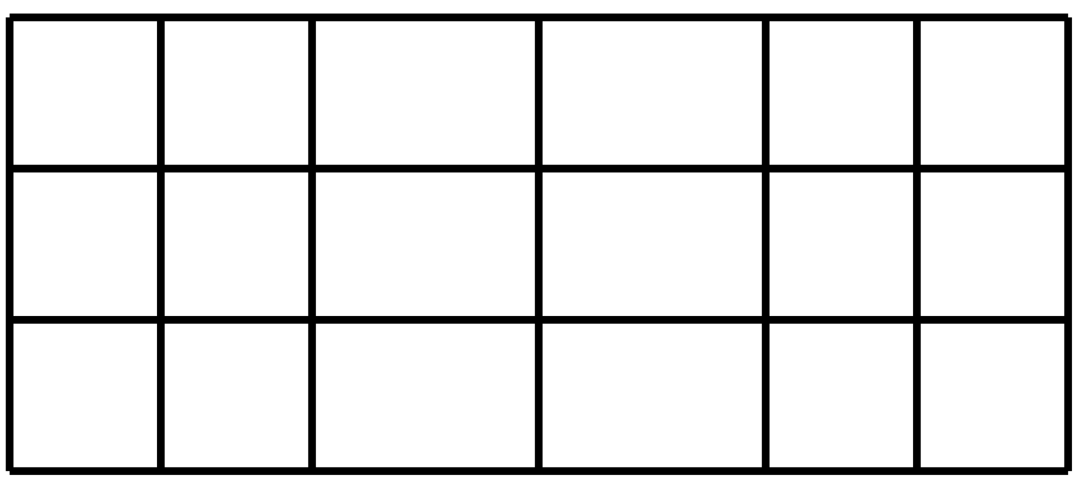
External forces are balanced with internal forces

PDE methods

- Example

$\nabla \nabla \cdot u$ is high

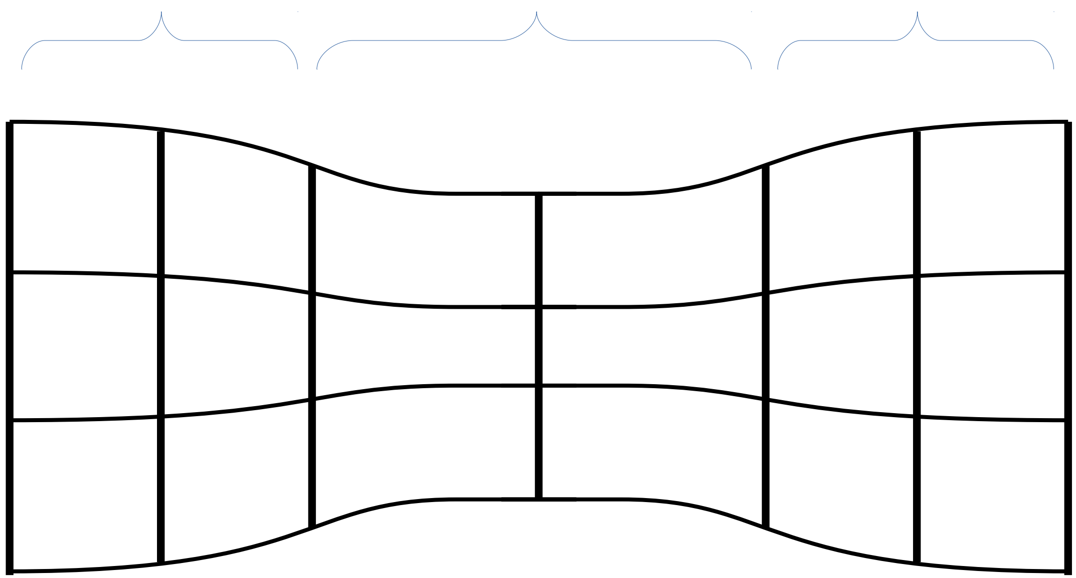
$\nabla \cdot u = 0 \quad \nabla \cdot u > 0 \quad \nabla \cdot u = 0$



PDE methods

- Example

$$\nabla \cdot u = 0 \qquad \nabla \cdot u = 0 \qquad \nabla \cdot u = 0$$



PDE methods

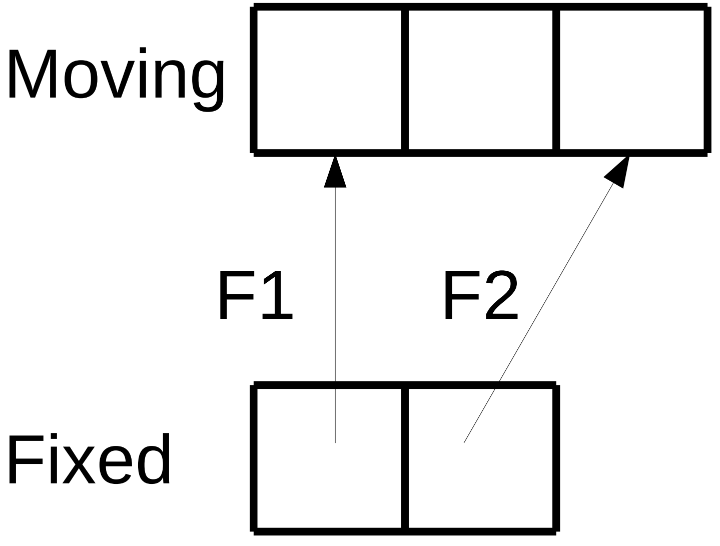
- Linear elastic equation

$$\mu \nabla^2 u + (\mu + \lambda) \nabla \nabla \cdot u + F = 0$$

Lamé's parameters

PDE methods

- External Forces

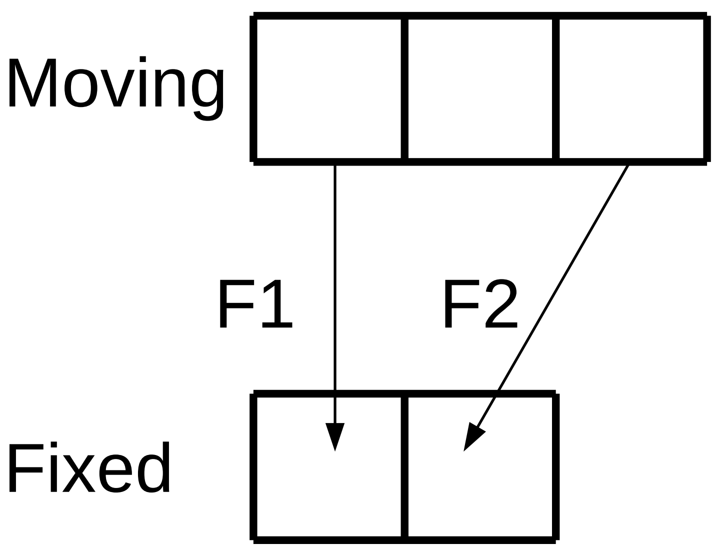


Forces defined by similarity metric
(e.g. SSD)

Zero force on F1
Negative force on F2
(for moving image)

PDE methods

- External Forces



Forces defined by similarity metric
(e.g. SSD)

Zero force on F1
Negative force on F2
(for moving image)

PDE methods

- Relation between PDE and regularization penalty

	Regularization Penalty	PDE
Linear Elastic	$\int \frac{\mu}{4} \sum_{j,k} (\partial_{x_j} u_k + \partial_{x_x} u_j)^2 + \frac{\lambda}{2} (\nabla \cdot u)^2$	$\mu \nabla^2 u + (\mu + \lambda) \nabla \nabla \cdot u + F = 0$
Diffusion	$\int \frac{1}{2} \sum_j \|\nabla u_j\|^2$	$\nabla^2 u + F = 0$
Curvature	$\int \frac{1}{2} \sum_j \|\nabla^2 u_j\|^2$	$\nabla^4 u + F = 0$

PDE Algorithm

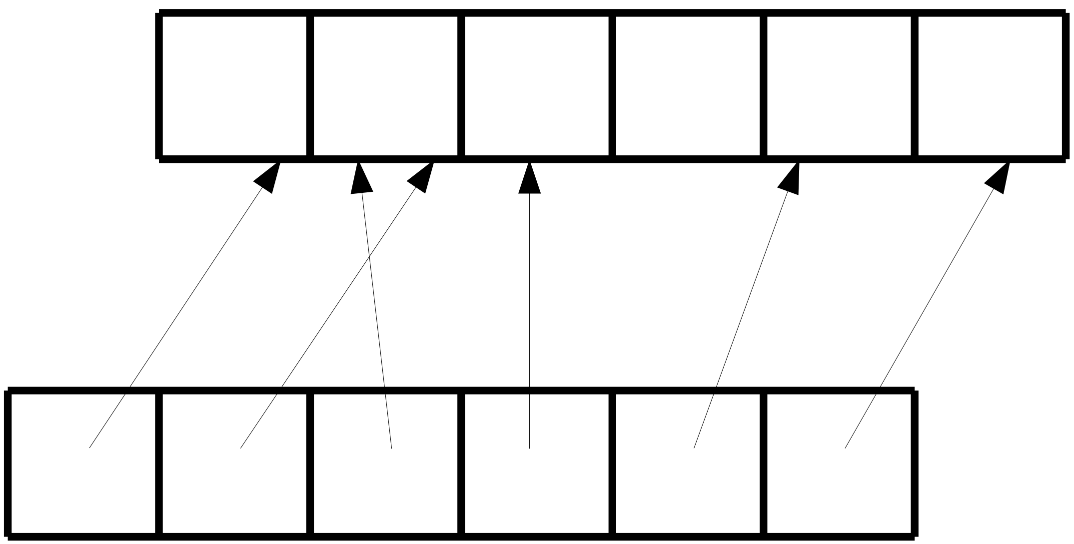
- For each iteration k
  - Compute forces, e.g.

$$F_k = (f - m \circ t)\nabla m$$

  - Solve PDE
    - SOR, Multigrid, etc

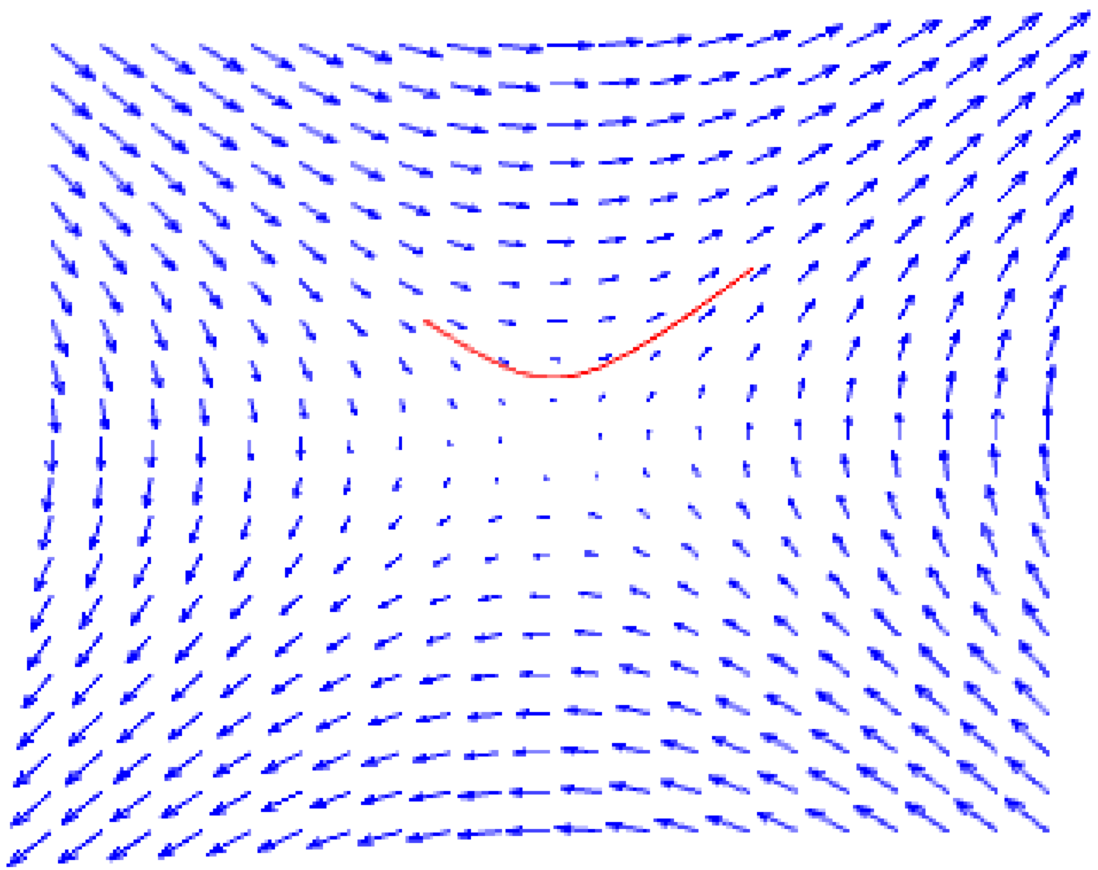
Diffeomorphic methods

- Diffeomorphic: smooth with smooth inverse
- This result is not diffeomorphic



Diffeomorphic methods

- Define a velocity field
- Transformation is integral of velocity field



Diffeomorphic methods

- Let τ be time
- Let $v(x)$ be a (stationary) velocity field
- Then

$$v(x) = \frac{d\,u(x)}{d\tau}$$

- But *x* varies over time, so:

$$v(t(x,\tau)) = \frac{d\,t(x,\tau)}{d\tau}$$

Diffeomorphic methods

- The solution to the ODE is an exponential

$$u = \exp(v)$$

Diffeomorphic methods

- Exponentiation algorithm (Vercauteren 08)
- Based on property

$$\exp((a+b)t)=\exp(at)\circ\exp(bt)$$

1. Choose N such that $2^{-N}v$ is small
2. Take single integration step $u \leftarrow 2^{-N}v$
3. Perform N recursive compositions $u \leftarrow u \circ u$

Diffeomorphic demons

- Initialize *v* to zero
- For each iteration *k*
  - Compute update field on *v*

$$v_k = v_{k-1} + \frac{(f - m \circ t)\nabla m}{(f - m \circ t)^2 + \|\nabla m\|^2}$$

  - Compute $u_k = \exp(v_k)$
  - Smooth *u* with Gaussian

Fluid methods

- Linear elastic and diffusion regularizers penalize curvature of displacement field (or velocity field)
- Fluid methods allow large deformations by removing this penalty

Fluid demons

- Initialize *u* to zero
- For each iteration *k*
  - Compute update field on *u*

$$d_k = \frac{(f - m \circ t)\nabla m}{(f - m \circ t)^2 + \|\nabla m\|^2}$$

  - Smooth *d* with Gaussian
  - $u_k = u_{k-1} + d_k$

What did we learn?

- Review of multivariate calculus
- PDE methods
- Diffeomorphic methods
- Fluid methods