MASSACHUSETTS
GENERAL HOSPITAL

RADIATION ONCOLOGY

Deformable Image Registration
Part 1

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Preliminaries

- Transformation is defined on *fixed image*
  - A.k.a. Reference image, static image
- Transformation maps fixed image to *moving image*
  - A.k.a. Test image, target image

Preliminaries

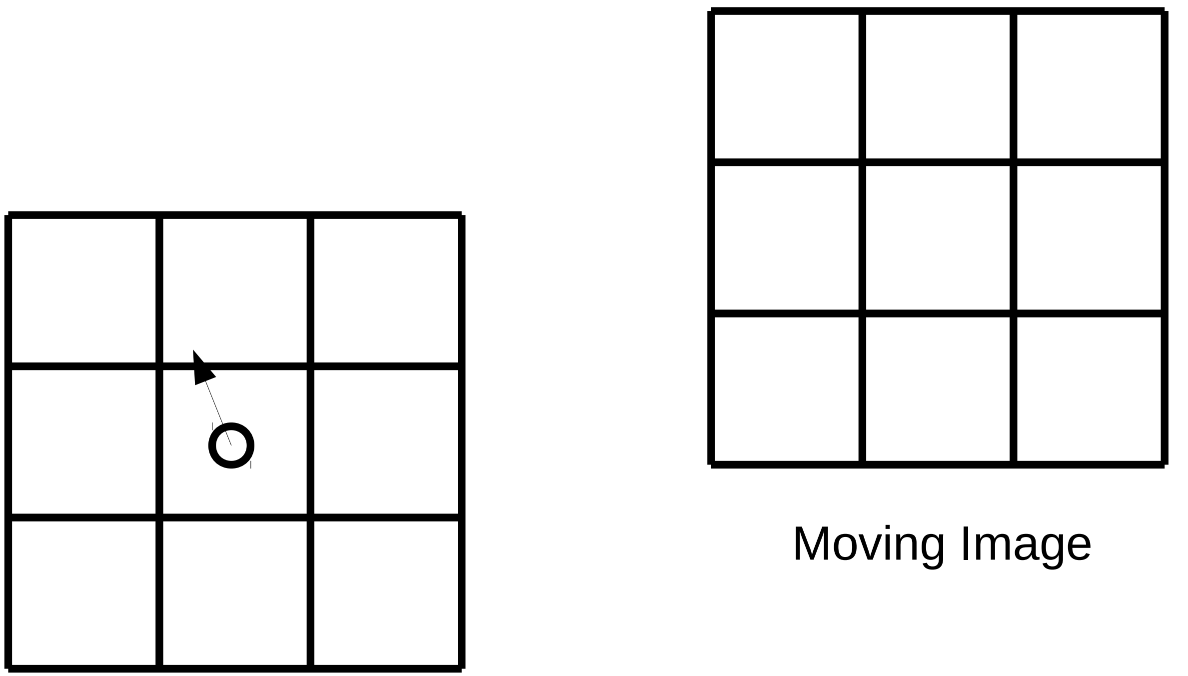
- Transform vs. Displacement Field
  - “Vector field” or “Deformation field” usually refers to displacement field

$$t(x) = x + u(x)$$

  - *t* is the transform
  - *u* is the displacement

Preliminaries

$$u(x) = [-0.2, 0.6]$$

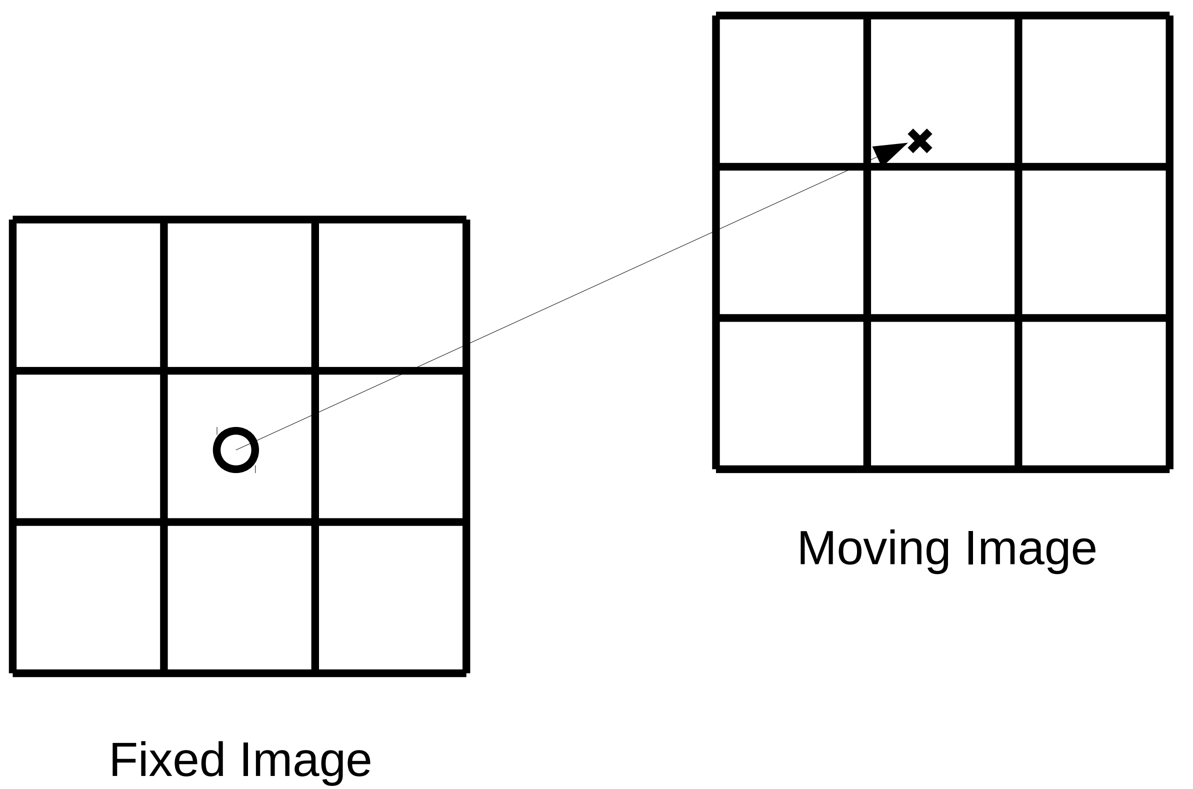


Fixed Image

Moving Image

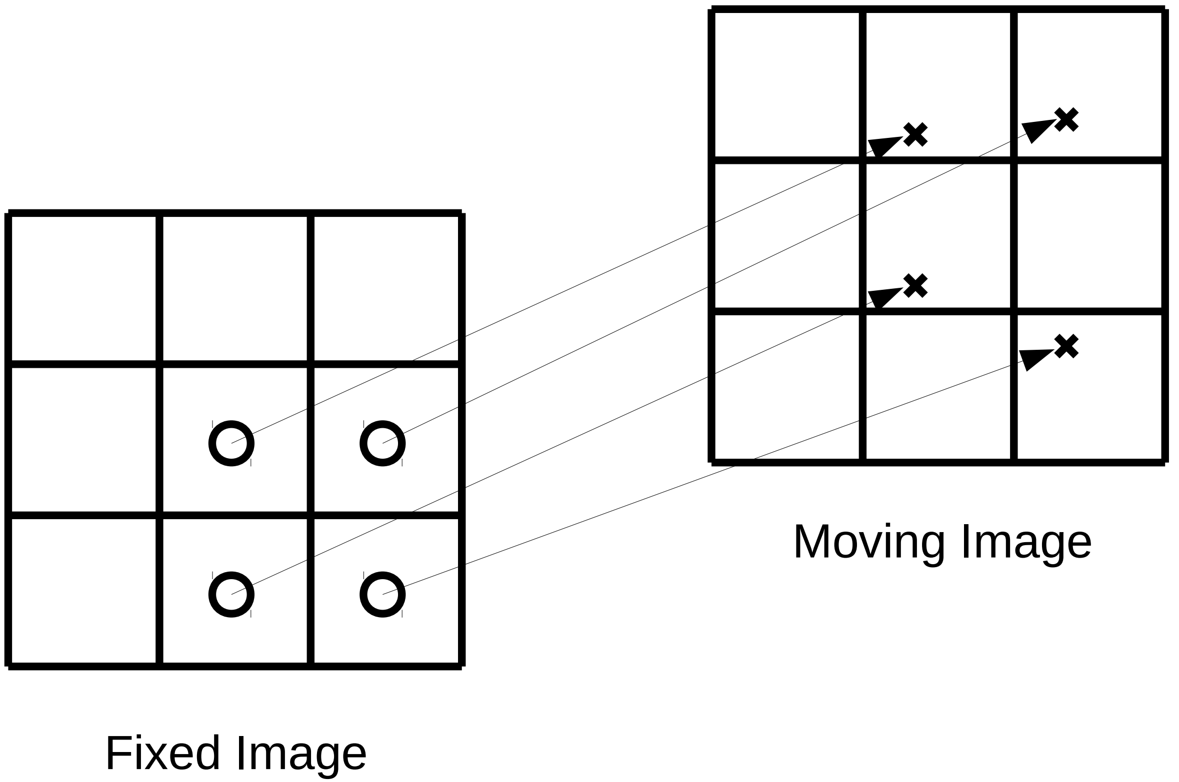
Preliminaries

$$t(x) = [0, 0] + [-0.2, 0.6]$$



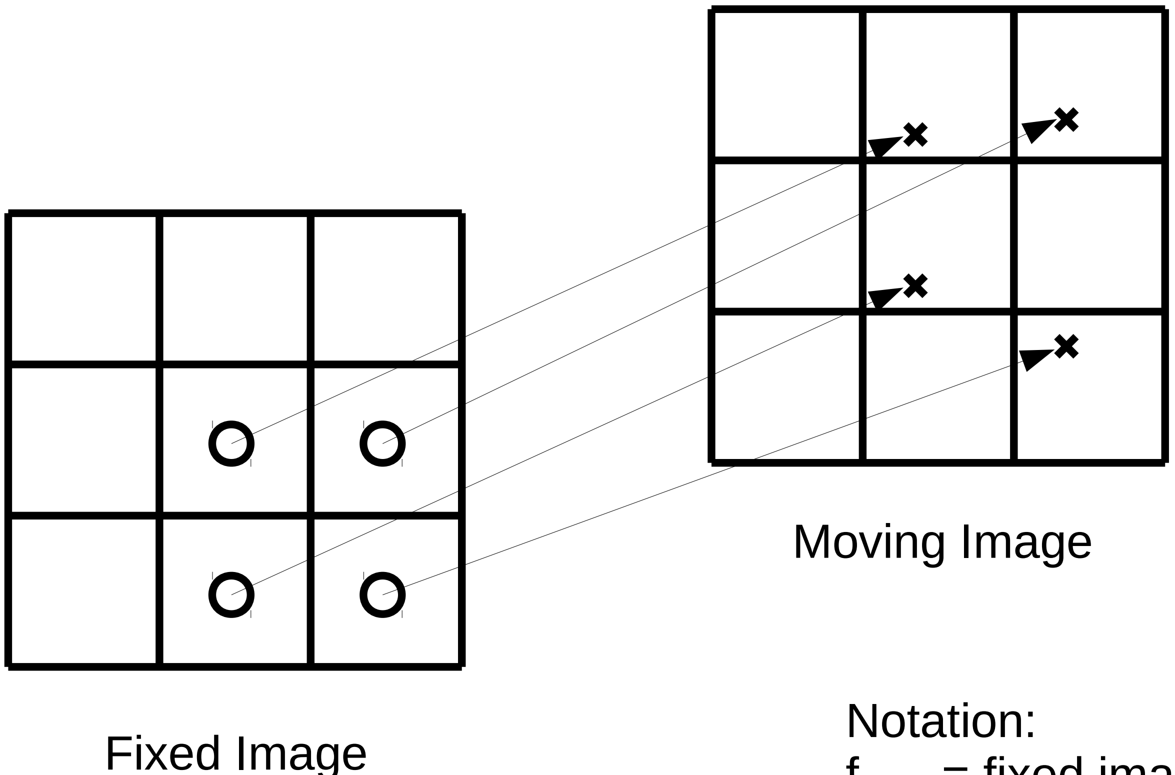
Preliminaries

- Image warping by “pulling”



Preliminaries

$$w(x)=m(t(x))$$

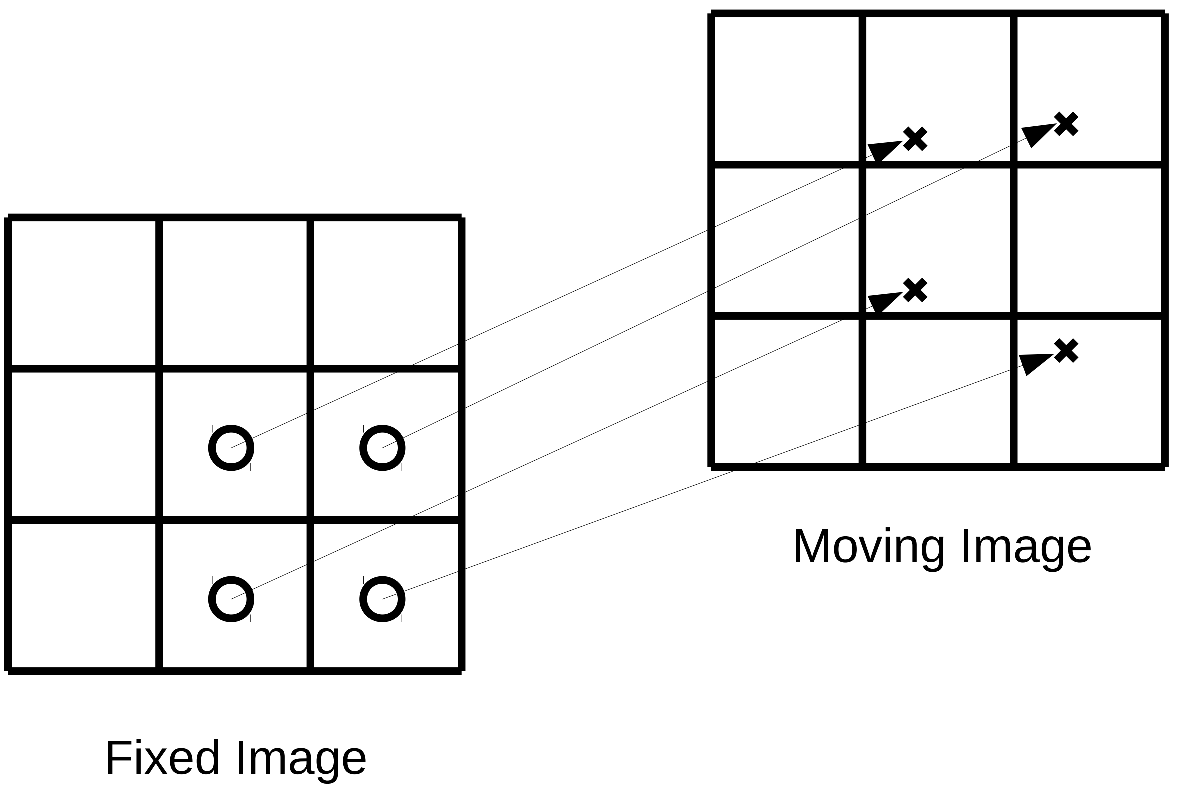


Notation:
f = fixed image intensity
m = moving image intensity
w = warped image intensity

Preliminaries

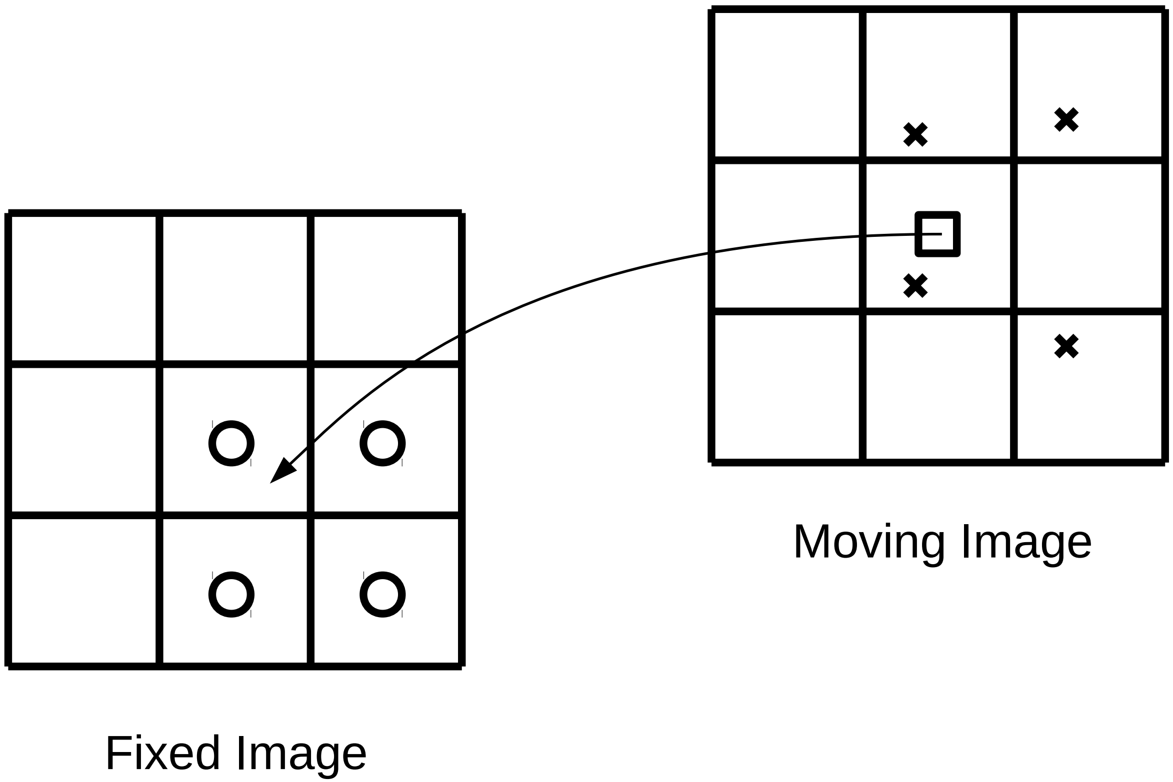
- Could we "push"?

$$w(x) = f(t^{-1}(x))$$



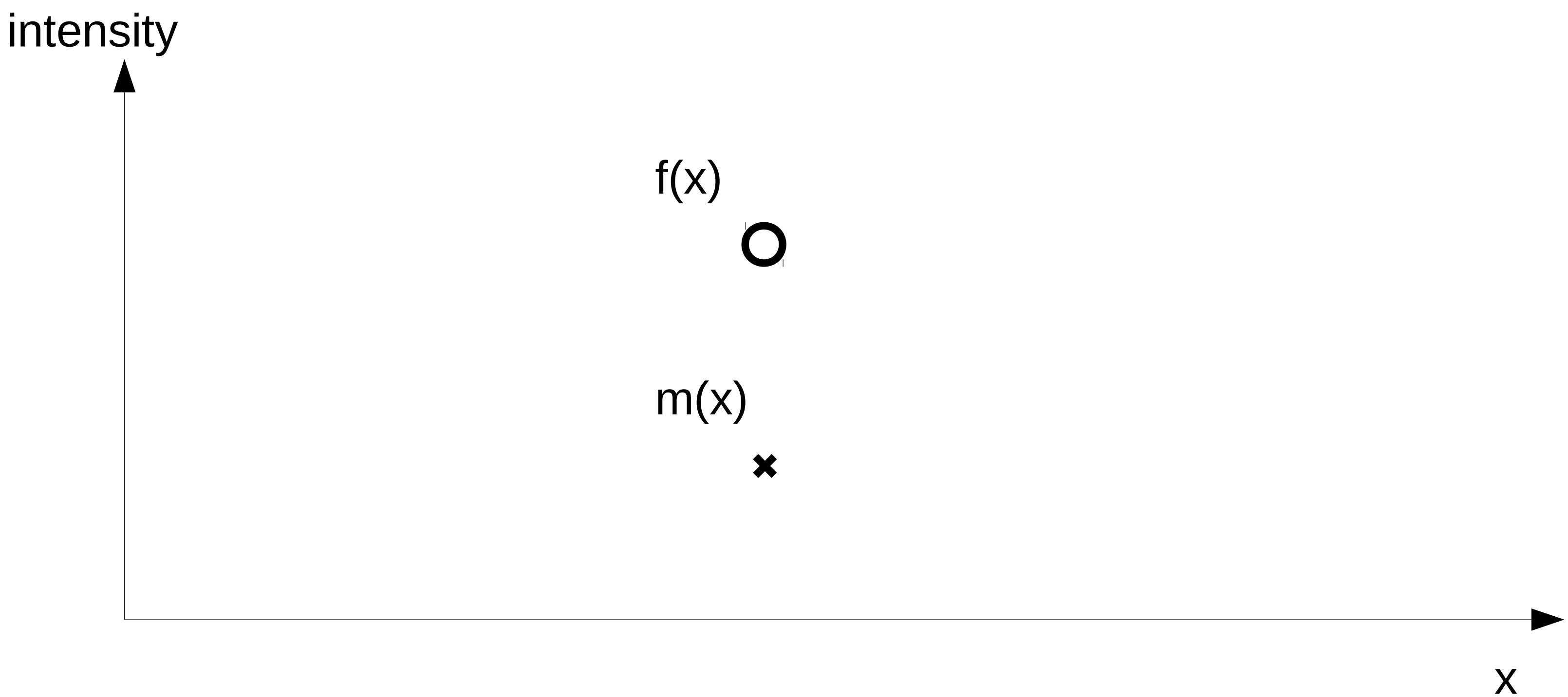
Preliminaries

- Could we “search”? $w(x)=f(t^{-1}(x))$



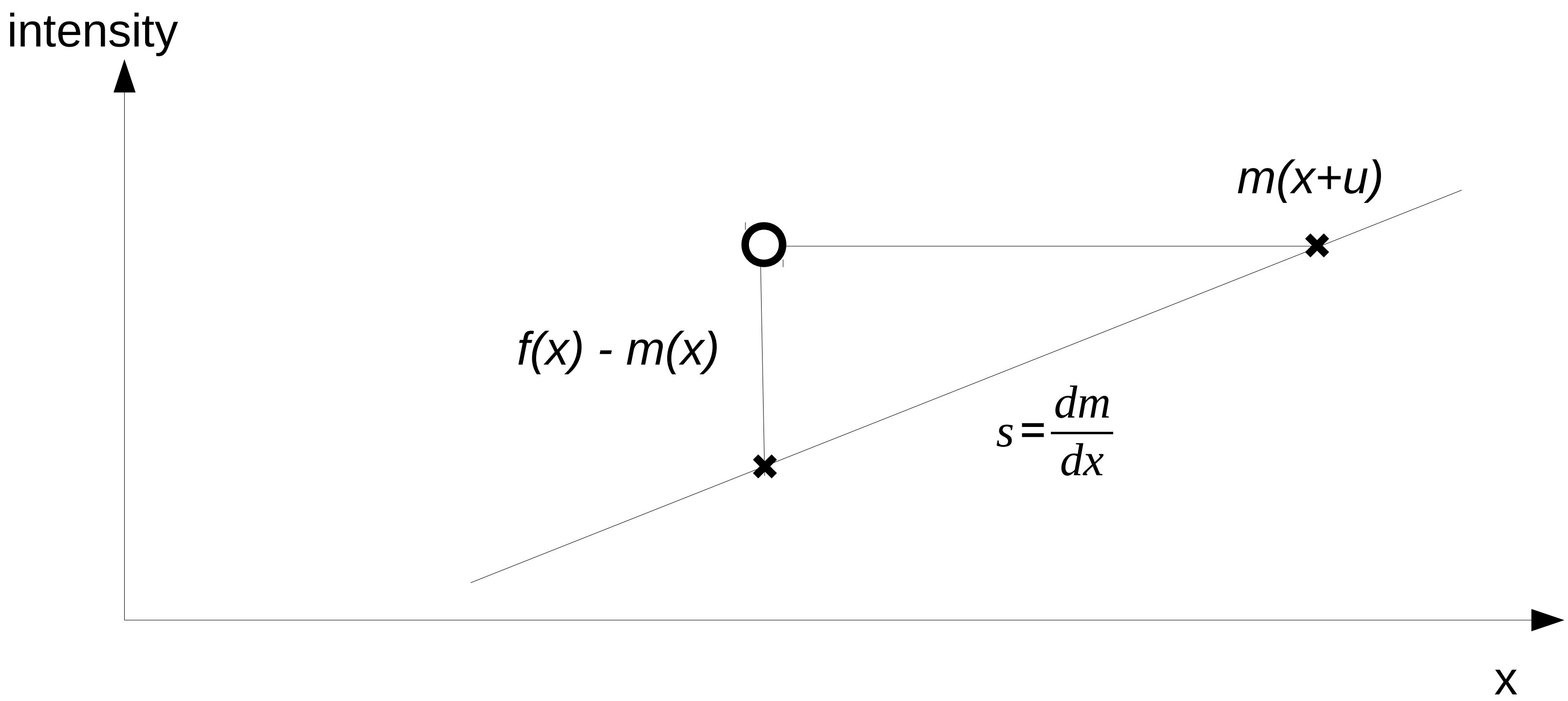
Quadratic Error Function

- Consider a single voxel in 1-D



Quadratic Error Function

- If *m* is a linear function, we can compute *u*



Quadratic Error Function

- “SSD” or “MSE” error function for a single voxel

$$
\begin{aligned}
C(u) &= \|f(x)-m(x+u)\|^2 \\
&= \|f(x)-(m(x)+u\,s)\|^2 \\
&= \|(f(x)-m(x))-u\,s\|^2 \\
&= \|e-u\,s\|^2
\end{aligned}
$$

Quadratic Error Function

- Calculate derivative

$$\frac{dC}{du} = \frac{d}{du} \| e - u s \|^2$$

$$= -2 e s + 2 u s^2$$

- Setting derivative to zero yields

$$\Rightarrow u = e / s$$

Quadratic Error Function

- In one dimension

$$u(x)=\frac{f(x)-m(x)}{dm/dx}$$

- In two or three dimensions

$$u(x) \quad = \quad \frac{f(x)-m(x)}{\nabla m}$$

$$\rightarrow \quad \frac{(f(x)-m(x))\nabla m}{\|\nabla m\|^2}$$

Notational Convenience

- We'll drop the spatial location *x* when possible

$$u = \frac{(f - m)\nabla m}{\|\nabla m\|^2}$$

- Use operator notation for image warping

$$m \circ t(x) \stackrel{\text{def}}{=} m(t(x))$$

DIR Algorithm “A-1”

- Let

$$u = \frac{(f - m)\nabla m}{\|\nabla m\|^2}$$

Is this a good algorithm?

DIR Algorithm “A-2”

- Let $u = 0$
- For each iteration k

$$u_k = u_{k-1} + \frac{(f - m \circ t)\nabla m}{\|\nabla m\|^2}$$

Is this a good algorithm?

DIR Algorithm “B-1”

- For each *x* in domain of *f*
  - Find *u* that minimizes

$$\| f(x) - m(x+u) \|^2$$

Is this a good algorithm?

DIR Algorithm “B-2”

- For each *x* in domain of *f*
  - Find *u* that minimizes

$$\|f(x) - m(x+u)\|^2$$

subject to

$$\|u\|^2 < \omega$$

Is this a good algorithm?

Demons algorithm

- Stabilized step size

$$u_k = u_{k-1} + \frac{(f - m \circ t)\nabla m}{(f - m \circ t)^2 + \|\nabla m\|^2}$$

The denominator can still be unstable, but only when $f \approx m \circ t$ and $\nabla m \approx 0$

DIR Algorithm “Demons”

- Let $u = 0$
- For each iteration k
  - Let $$u_k = u_{k-1} + \frac{(f - m \circ t)\nabla m}{(f - m \circ t)^2 + \|\nabla m\|^2}$$
  - Smooth u with Gaussian filter

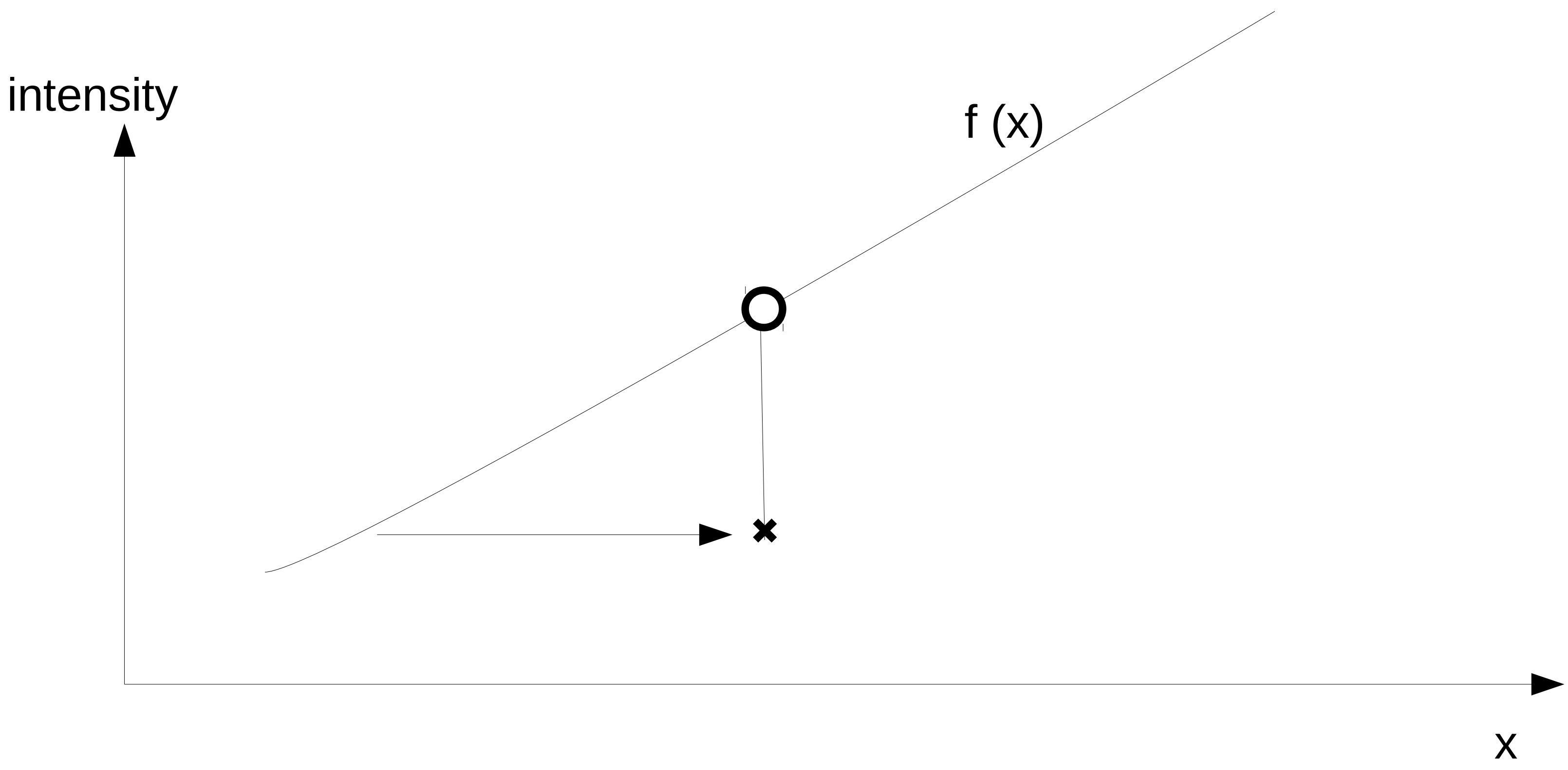
What are the salient features of this algorithm?

DIR Algorithm “Demons”

- Some salient features
  - Easy to implement
  - Each iteration is fast
  - Many iterations may be required
    - Small step size
    - Smoothing operation acts as diffusion
  - Very flexible
    - Step size and smoother can be modified
  - Difficult to understand the metric being minimized

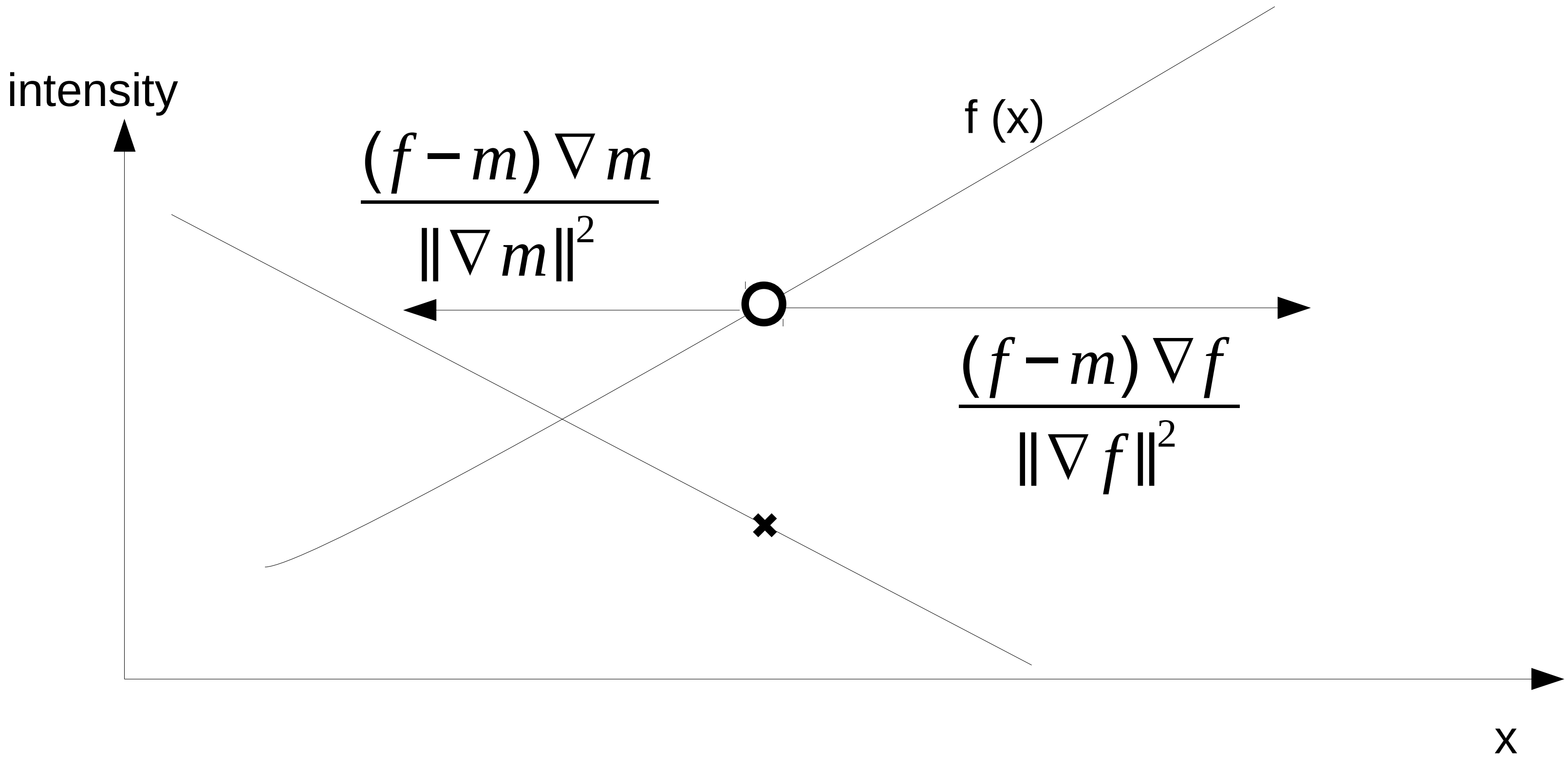
Symmetrized Step

- Why not use gradient of *f* ?



Symmetrized Step

- Could be useful when dm/dx ~ 0
  or df/dx does not agree with dm/dx



Vector Field Jacobian

- In order to achieve a symmetrized step, we must use the Jacobian.
- Let $t=[t_x, t_y, t_z]$
- The Jacobian of *t* is

$$Jac(t)=\begin{bmatrix} \frac{dt_x}{dx} & \frac{dt_x}{dy} & \frac{dt_x}{dz} \\ \frac{dt_y}{dx} & \frac{dt_y}{dy} & \frac{dt_y}{dz} \\ \frac{dt_z}{dx} & \frac{dt_z}{dy} & \frac{dt_z}{dz} \end{bmatrix}$$

Vector Field Jacobian

- Let us investigate a 2D example.
  The center of the grid is location (0,0).
  The current value of *t* is [-y, x]

f

-1	0	1
-1	0	1
-1	0	1

m

0	0	0
-1	-1	-1
-2	-2	-2

Vector Field Jacobian

-1	0	1
-1	0	1
-1	0	1

f

1,0	1,0	1,0
1,0	1,0	1,0
1,0	1,0	1,0

∇f

0	0	0
-1	-1	-1
-2	-2	-2

m

-1,-1	-1,0	-1,1
0,-1	0,0	0,1
1,-1	1,0	1,1

$t = [-y, x]$

-2	-1	0
-2	-1	0
-2	-1	0

$m \circ t$

Vector Field Jacobian

- At location (0,0), we have

$$\frac{(f - m \circ t)\nabla f}{\|\nabla f\|^2} = \frac{(0-(-1))[1,0]}{1} = [1,0]$$

- This is the correct direction in $m \circ t$ but not *m*

Vector Field Jacobian

- To find the direction in *m*, we calculate in the coordinate system of *m*

$$\frac{(f-m\circ t)\nabla f}{\|\nabla f\|^2}\rightarrow\frac{(f\circ t^{-1}-m)\nabla(f\circ t^{-1})}{\|\nabla(f\circ t^{-1})\|^2}$$

- By the chain rule, we have

$$\nabla(f\circ t^{-1})=\nabla f\circ Jac(t^{-1})$$

- And by the inverse function theorem

$$Jac(t^{-1})=Jac^{-1}(t)$$

Vector Field Jacobian

- Therefore, for t = [-y, x]

$$Jac(t) = \begin{bmatrix} \frac{dt_x}{dx} & \frac{dt_x}{dy} \\ \frac{dt_y}{dx} & \frac{dt_y}{dy} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Jac(t^{-1}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\nabla(f \circ t^{-1}) = [1,0]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [0,1]$$

Vector Field Jacobian

-1	0	1
-1	0	1
-1	0	1

$$f$$

1,0	1,0	1,0
1,0	1,0	1,0
1,0	1,0	1,0

$$\nabla f$$

0,1	0,1	0,1
0,1	0,1	0,1
0,1	0,1	0,1

$$\nabla f \circ Jac(t^{-1})$$

0	0	0
-1	-1	-1
-2	-2	-2

$$m$$

-1,-1	-1,0	-1,1
0,-1	0,0	0,1
1,-1	1,0	1,1

$$t=[-y, x]$$

-2	-1	0
-2	-1	0
-2	-1	0

$$m \circ t$$

What have we learned?

- Warping based on “pulling”
- Quadratic image cost function (SSD)
- Use of image gradient to find correspondences
- Stabilized step length
- Demons algorithm
- How to use Jacobian matrix