Image Reconstruction 2b – Fully 3D Reconstruction

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The 3D Radon transform and its inverse
- The 3D Radon transform
- Inversion of the 3D Radon transform

Getting 3D Radon transform from cone beam data (1990s)
- Grangeat’s trick

Tuy theorem

Helical scanning

The Katsevich breakthrough (2002)
The 3D Radon transform of \( f(\mathbf{x}) \) is the integral of \( f(\mathbf{x}) \) over 2D planes perpendicular to \( \hat{\mathbf{n}}_\Omega \):

\[
\mathcal{R} f(p, \hat{\mathbf{n}}_\Omega) = \int_V f(\mathbf{x}) \delta(p - \mathbf{x} \cdot \hat{\mathbf{n}}_\Omega) \, d^3x
\]

1. Note: The \( \delta \)-function “picks” those points \( \mathbf{x} \) that lie on the plane shown (plane at distance \( p \) from origin).

2. Note: In 3D (and higher dimensions) the Radon transform differs from the “x-ray transform” (integration over lines).
The inverse 3D Radon transform

Inversion of the Radon transform in 3D is beautifully simple:

\[
f(x) = -\frac{1}{8\pi^2} \int_{4\pi} \mathcal{R}'' f(x \cdot \hat{n}_\Omega, \hat{n}_\Omega) \, d\Omega,
\]

where

\[
\mathcal{R}'' f(p, \hat{n}_\Omega) = \frac{\partial^2 \mathcal{R}'' f(p, \hat{n}_\Omega)}{\partial p^2}.
\]
The inverse 3D Radon transform

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Steps involved:

1. take second derivative (with respect to \( p \)) of Radon transform,
2. back-project over planes containing \( \mathbf{x} \),
3. integrate over all angles.
Intuitive "derivation" of inverse 3D Radon transform

- Central slice theorem in 3D: every back-projection ("smearing out" over planes) for a given orientation $\hat{n}_\Omega$ corresponds with adding a line through the origin of the Fourier domain.
- The density of lines falls off as $1/\rho^2$.
- To compensate for the low-pass effect, we must multiply with $\rho^2$ (compare with the $|\rho| = |\nu|$ filter in the 2D case).
- Multiplication with $\rho^2$ in the Fourier domain corresponds with taking the second derivative in the spatial domain.
Intuitive "derivation" of inverse 3D Radon transform

Fourier domain: $F(\rho)$

Back-projection means adding lines through the origin of the Fourier domain.

Density of lines falls off as $1/|\rho|^2$ from the origin.
So what is the problem?

- The problem is that with x-rays we can only measure line integrals (along the rays), not the planar integrals needed for the 3D Radon transform.
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The problem is that with x-rays we can only measure line integrals (along the rays), not the planar integrals needed for the 3D Radon transform.

But, can’t we just integrate line integrals across the plane to get the planar integral?
Getting 3D Radon transform from cone beam data (1990s)

Getting 3D Radon transform from cone beam data

Top view

Side view

dp = L \cos \gamma \, d\alpha
Getting 3D Radon transform from cone beam data

Measured line integrals:

\[ \int_{0}^{\infty} f(x(L, \gamma, \alpha)) \, dL \]

Integrate over the plane:

\[ \int_{-\pi}^{\pi} \int_{0}^{\infty} f(x(L, \gamma, \alpha)) \, dL \, d\gamma = \int_{-\pi}^{\pi} \int_{0}^{\infty} \frac{1}{L} f(x(L, \gamma, \alpha)) \, L \, dL \, d\gamma \]

\[ \neq Rf(p, \hat{n}_\Omega) \]

This would give the integral of \( \frac{1}{L} f \) over the plane.
Grangeat’s trick

\[ dp = L \cos \gamma \, d\alpha \]
Grangeat’s "trick"

Grangeat’s idea: Compensate for incorrect integration over the plane by taking an incorrect derivative of the Radon transform with respect to $p$.

Apply a weight factor of $1/\cos \gamma$ and take the derivative with respect to the tilt angle, $\alpha$:

$$
\int_{-\pi}^{\pi} \frac{\partial}{\partial \alpha} \frac{1}{\cos \gamma} \int_{0}^{\infty} f(x(L, \gamma, \alpha)) \, dL \, d\gamma = \int_{-\pi}^{\pi} \frac{\partial}{\partial p} \frac{L \cos \gamma}{\cos \gamma} \int_{0}^{\infty} f(x(L, \gamma, \alpha)) \, dL \, d\gamma
$$

$$
= \frac{\partial}{\partial p} \int_{-\pi}^{\pi} \int_{0}^{\infty} f(x(L, \gamma, p)) \, L \, dL \, d\gamma
$$

$$
= \frac{\partial}{\partial p} \mathcal{R} f(p, \hat{n}_\Omega).
$$
Problems with Grangeat-type methods

- “Long object” problem – solvable, approximately
- Numerical instabilities
Tuy’s sufficiency condition

**Theorem (Tuy)**

*Any plane through an object point \( x \) must cut the source trajectory.*

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- Condition for trajectory to facilitate exact reconstruction

Examples of source trajectories – which ones fulfill the Tuy condition?
Helical scanning geometry
Helical scanning geometry: Resort into planes

Tube rotates around patient and moves along z-axis during helical acquisition

Nutating slice plane
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The Katsevich breakthrough (2002)

- Filtered backprojection algorithm for cone beam data
- Exact reconstruction (aside from discretization)
- Good for “long objects” (projection data only needed near the ROI)
The Katsevich algorithm

Detector signal and its derivative:

\[ D(y(s), \theta) \]
\[ D'(y(s), \theta) = \left. \frac{\partial D(y(q), \theta)}{\partial q} \right|_{q=s} \]

Further definitions:

\[ b = \frac{x - y(s)}{|x - y(s)|} ; \quad y(s): \text{ source position} \]
\[ e: \text{ perpendicular to } b, \text{ must be appropriately chosen} \]
\[ b \text{ and } e \text{ span the } \kappa \text{ plane for filtering} \]
The Katsevich breakthrough (2002)

The Kappa plane

\[
x \quad y(s_{\kappa 1}) \quad y(s_{\kappa 2})
\]
The Kappa plane
The Katsevich algorithm

\[ f(x) = -\frac{1}{2\pi^2} \int_{I_{PI}(x)} \frac{I(s, x)}{|x - y(s)|} \, ds \]

where

\[ I(s, x) = \int_{-\pi}^{\pi} \frac{1}{\sin \gamma} D'(y(s), \cos \gamma b + \sin \gamma e) \, d\gamma \]

Note:

- \( \frac{1}{\sin \gamma} \) is the filter, it corresponds to a “Hilbert” filter.
- Backprojection is done over the PI (parametric interval), see Danielsson et al.
Results: 3D Shepp phantom – from Katsevich 2002
Results: Defrise slice phantom – from Katsevich 2002
Further reading on fully 3D reconstruction


- Papers by P.-E. Danielsson et al., Linköping, Sweden
